Positivity-preserving two-dimensional cell-centered Lagrangian schemes

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Eulerian formalism (spatial description)

- Fixed referential attached to the observer
- Fixed observation area in which the fluid flows through

Lagrangian formalism (material description)

- Moving referential attached to the material
- Observation area getting moved and deformed through the fluid flow

Advantages of the Lagrangian formalism

- Adapted to the study of regions undergoing large shape changes
- Naturally tracks interfaces in multimaterial compressible flows
- No numerical diffusion from the discretization of the convection terms

Disadvantages of the Lagrangian formalism

- **Robustness issue in cases of shear flows or vortexes**

$\Longrightarrow$ ALE (Arbitrary Lagrangian-Eulerian) method

Definitions

- ρ is the fluid density
- $\boldsymbol{u} = (u_1, u_2)^{\mathrm{t}}$ is the fluid velocity
- e is the fluid specific total energy
- p is the fluid pressure
- $\varepsilon = e - \frac{1}{2}\boldsymbol{u}^2$ is the fluid specific internal energy

Euler equations

- $\dfrac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho\, \boldsymbol{u}) = 0$ Continuity equation
- $\dfrac{\partial \rho\, \boldsymbol{u}}{\partial t} + \nabla_x \cdot (\rho\, \boldsymbol{u} \otimes \boldsymbol{u} + p\, \mathrm{I}_d) = \mathbf{0}$ Momentum conservation equation
- $\dfrac{\partial \rho\, e}{\partial t} + \nabla_x \cdot (\rho\, \boldsymbol{u}\, e + p\, \boldsymbol{u}) = 0$ Total energy conservation equation

Thermodynamical closure

- $p = p(\rho, \varepsilon)$ Equation of state (EOS)

Trajectory equation

- $\dfrac{\partial\,\boldsymbol{x}(\boldsymbol{X},t)}{\partial t} = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{X},t),t), \qquad \boldsymbol{x}(\boldsymbol{X},0) = \boldsymbol{X}$

Material time derivative

- $\varphi(\boldsymbol{x},t)$ is a fluid variable with sufficient smoothness
- $\dfrac{\mathrm{d}\,\varphi}{\mathrm{d}t} \equiv \dfrac{\partial\,\varphi(\boldsymbol{x}(\boldsymbol{X},t),t)}{\partial t} = \dfrac{\partial\,\varphi}{\partial t} + \boldsymbol{u}\,\boldsymbol{.}\,\nabla_x\varphi$

Lagrangian equations

- $\rho\,\dfrac{\mathrm{d}\,(1/\rho)}{\mathrm{d}t} - \nabla_x\,\boldsymbol{.}\,\boldsymbol{u} = 0$
- $\rho\,\dfrac{\mathrm{d}\,\boldsymbol{u}}{\mathrm{d}t} + \nabla_x p = \boldsymbol{0}$
- $\rho\,\dfrac{\mathrm{d}\,e}{\mathrm{d}t} + \nabla_x\,\boldsymbol{.}\,(p\,\boldsymbol{u}) = 0$

Definitions

- $\tau = \frac{1}{\rho}$ is the specific volume
- $\mathrm{U} = (\tau, u_1, u_2, e)^{\mathrm{t}}$ is the variables vector
- $\mathrm{F}(\mathrm{U}) = (-\boldsymbol{u}, p\,\mathbb{1}(1), p\,\mathbb{1}(2), p\,\boldsymbol{u})^{\mathrm{t}}$ is the flux vector
- $\mathbb{1}(i) = (\delta_{i1}, \delta_{i2})^{\mathrm{t}}$

Updated Lagrangian formulation

- $\rho \dfrac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \nabla_x \,.\, \mathrm{F}(\mathrm{U}) = 0$ Moving configuration

Integral conservative form

- $\dfrac{\partial}{\partial t} \displaystyle\int_\omega \rho\,\mathrm{U}\,\mathrm{d}v + \int_{\partial\omega} \mathrm{F}(\mathrm{U}) \,.\, \boldsymbol{n}\,\mathrm{d}s = 0$ Moving configuration

Thermodynamical closure

- $p = p(\rho, \varepsilon)$ Equation of state (EOS)

Ideal EOS for the perfect gas

- $p = \rho\,(\gamma - 1)\,\varepsilon$ where $a = \sqrt{\frac{\gamma\,p}{\rho}}$
- If $\rho > 0$ then $\varepsilon > 0 \iff a^2 > 0 \quad (\Leftrightarrow\ p > 0)$

Stiffened EOS for water

- $p = \rho\,(\gamma - 1)\,\varepsilon - \gamma\,p_s$ where $a = \sqrt{\frac{\gamma\,(p+p_s)}{\rho}}$
- If $\rho > 0$ then $\rho\varepsilon > p_s \iff a^2 > 0 \quad (\Leftrightarrow\ p > -p_s)$

Jones-Wilkins-Lee (JWL) EOS for the detonation-products gas

- $p = \rho\,(\gamma - 1)\,\varepsilon + f_j(\rho)$ where $a = \sqrt{\frac{\gamma\,p - f_j(\rho) + \rho\,f_j'(\rho)}{\rho}}$
- If $\rho > 0$ then $\varepsilon > 0 \implies a^2 > 0 \quad (\Leftrightarrow\ p > f_j(\rho) \geq 0)$

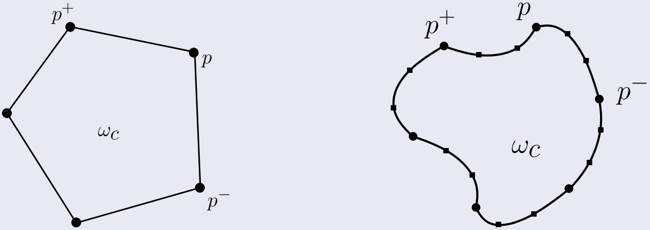
Mie-Grüneisen EOS for solids

- $p = \rho_0\,\Gamma_0\,\varepsilon + \rho_0\,a_0^2\,f_m(\rho)$ where $a = \sqrt{a_0^2\,f_m'(\eta) + \frac{\rho_0\,\Gamma_0\,p}{\rho^2}}$
- If $\rho \in [\rho^\star, \frac{S_m}{S_m - 1}\,\rho_0[$ then $\varepsilon > 0 \implies a^2 > 0\ (\Leftrightarrow\ p > \rho_0\,a_0^2\,f_m(\rho))$

Definitions

- $0 = t^0 < t^1 < \cdots < t^N = T$ is a partition of the time domain $[0, T]$
- $\Delta t^n = t^{n+1} - t^n$ is the n^{th} time step
- $\omega = \bigcup_c \omega_c$ is a partition of the moving domain ω
- $|\omega_c|$ is the volume of cell ω_c
- $m_c = \rho_c^n \, |\omega_c|$ is the constant mass of cell ω_c

Generic cell



(a) Polygonal cell. (b) Curved polygonal cell.

- $\mathcal{Q}_c$ is a control point set of cell ω_c

Integral conservative system of equations

- $\dfrac{\partial}{\partial t}\displaystyle\int_{\omega} \rho\, \mathrm{U}\, \mathrm{d}v + \int_{\partial\omega} \mathrm{F}(\mathrm{U}) \cdot \boldsymbol{n}\, \mathrm{d}s = 0$ Moving configuration

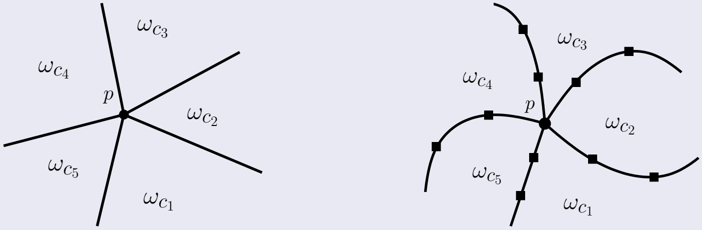
First-order finite volume scheme

- $\mathrm{U}_c^n = \dfrac{1}{m_c}\displaystyle\int_{\omega_c} \rho(\boldsymbol{x}, t^n)\, \mathrm{U}(\boldsymbol{x}, t^n)\, \mathrm{d}v$
- $\mathrm{U}_c^{n+1} = \mathrm{U}_c^n - \dfrac{\Delta t^n}{m_c} \displaystyle\sum_{q \in \mathcal{Q}_c} \overline{\mathrm{F}}_{qc} \cdot l_{qc} \boldsymbol{n}_{qc}$
- $\boldsymbol{x}_q^{n+1} = \boldsymbol{x}_q^n + \Delta t^n\, \overline{\boldsymbol{u}}_q$

Control point numerical fluxes

- $\overline{\mathrm{F}}_{qc} = (-\overline{\boldsymbol{u}}_q,\ \mathbb{1}(1)\, \overline{p}_{qc},\ \mathbb{1}(2)\, \overline{p}_{qc},\ \overline{p}_{qc}\, \overline{\boldsymbol{u}}_q)^{\mathrm{t}}$ Local to the cell
- $\overline{p}_{qc} = p_c^n - \widetilde{z}_{qc}\, (\overline{\boldsymbol{u}}_q - \boldsymbol{u}_c^n) \cdot \boldsymbol{n}_{qc}$ 1D approximate Riemann solver
- $\widetilde{z}_{qc} > 0$ Local approximation of the acoustic impedance

Node neighboring cells



(c) Polygonal cell.

(d) Curved polygonal cell.

- $\mathcal{C}_p$ is the neighboring cell set of node p

Local 1D approximate Riemann solver

- $\overline{p}_{qc} = p_c^n - \widetilde{z}_{qc}\,(\overline{\boldsymbol{u}}_q - \boldsymbol{u}_c^n)\,.\,\boldsymbol{n}_{qc}$ Loss of the conservation

Scheme conservation

- $\sum_c m_c\,\mathrm{U}_c^{n+1} = \sum_c m_c\,\mathrm{U}_c^n \quad \Longleftrightarrow \quad \sum_c \sum_{q\in\mathcal{Q}_c} \overline{p}_{qc}\, l_{qc}\boldsymbol{n}_{qc} = \boldsymbol{0}$

GLACE scheme (B. Després *et al*) 2005

- $\mathcal{Q}_c = \mathcal{P}_c$ is the node set of cell ω_c
- $\forall q \in \mathcal{Q}_c, \quad l_{qc}\boldsymbol{n}_{qc} = \frac{1}{2}(l_{q^-q}\boldsymbol{n}_{q^-q} + l_{qq^+}\boldsymbol{n}_{qq^+})$
- $\overline{\boldsymbol{u}}_p = \Big(\sum_{c\in\mathcal{C}_p} \mathrm{M}_{pc}\Big)^{-1} \sum_{c\in\mathcal{C}_p} \Big(\mathrm{M}_{pc}\boldsymbol{u}_c^n + p_c^n\, l_{pc}\boldsymbol{n}_{pc}\Big)$
- $\mathrm{M}_{pc} = \widetilde{z}_{pc}\, l_{pc}\,(\boldsymbol{n}_{pc} \otimes \boldsymbol{n}_{pc})$

EUCCLHYD scheme (P.-H. Maire *et al*) 2007

- $\mathcal{Q}_c = \bigcup_{p\in\mathcal{P}_c} \{p, p^+\}$ is the union of the face control point set $\mathcal{Q}(f_{pp^+})$
- $q \in \mathcal{Q}(f_{pp^+}) = \{p, p^+\}, \quad l_{qc}\boldsymbol{n}_{qc} = \frac{1}{2} l_{pp^+}\boldsymbol{n}_{pp^+}$
- $\overline{\boldsymbol{u}}_p = \Big(\sum_{c\in\mathcal{C}_p} \mathrm{M}_{pc}\Big)^{-1} \sum_{c\in\mathcal{C}_p} \Big(\mathrm{M}_{pc}\boldsymbol{u}_c^n + p_c^n\, l_{pc}\boldsymbol{n}_{pc}\Big)$
- $\mathrm{M}_{pc} = \widetilde{z}_{pc}^-\, l_{pc}^-\,(\boldsymbol{n}_{pc}^- \otimes \boldsymbol{n}_{pc}^-) + \widetilde{z}_{pc}^+\, l_{pc}^+\,(\boldsymbol{n}_{pc}^+ \otimes \boldsymbol{n}_{pc}^+)$

CCDG scheme (F. Vilar *et al*) 2012

- $\mathcal{Q}_c = \bigcup_{p\in\mathcal{P}_c} \mathcal{Q}(f_{pp^+})$
- $q \in \mathcal{Q}(f_{pp^+}), \quad l_{qc}\,\boldsymbol{n}_{qc} = \int_0^1 \lambda_q(\zeta) \sum_{k\in\mathcal{Q}(f_{pp^+})} \frac{\partial\lambda_k}{\partial\zeta}(\boldsymbol{x}_k \times \boldsymbol{e}_z)\,\mathrm{d}\zeta$
- $\overline{\boldsymbol{u}}_q = \left(\frac{\widetilde{z}_{qL}\,\boldsymbol{u}_L^n + \widetilde{z}_{qR}\,\boldsymbol{u}_R^n}{\widetilde{z}_{qL} + \widetilde{z}_{qR}} \right) - \frac{p_R^n - p_L^n}{\widetilde{z}_{qL} + \widetilde{z}_{qR}}\,\boldsymbol{n}_{qL} \qquad \forall q \in \mathcal{Q}_c \setminus \mathcal{P}_c$
- $\overline{\boldsymbol{u}}_p = \left(\sum_{c\in\mathcal{C}_p} \mathrm{M}_{pc} \right)^{-1} \sum_{c\in\mathcal{C}_p} \left(\mathrm{M}_{pc}\boldsymbol{u}_c^n + p_c^n\, l_{pc}\boldsymbol{n}_{pc} \right) \qquad \forall p \in \mathcal{P}_c$
- $\mathrm{M}_{pc} = \widetilde{z}_{pc}^-\, l_{pc}^-\,(\boldsymbol{n}_{pc}^- \otimes \boldsymbol{n}_{pc}^-) + \widetilde{z}_{pc}^+\, l_{pc}^+\,(\boldsymbol{n}_{pc}^+ \otimes \boldsymbol{n}_{pc}^+)$

Finite volume scheme on conical meshes (P. Hoch *et al*) 2011

- B. Boutin, E. Deriaz, P. Hoch and P. Navaro, *Extension of ALE methodology to unstructured conical meshes.* ESAIM: Proceedings, 32:32-55, 2011.

Requirements

- $|\omega_c| > 0 \iff \tau_c^n > 0$ Positive volume and density
- $(a_c^n)^2 = (a(\mathrm{U}_c^n))^2 > 0$ Computable sound speed

Convex admissible set

- $$G = \left\{ \mathrm{U} = \begin{pmatrix} \tau \\ \boldsymbol{u} \\ e \end{pmatrix}, \quad \tau \in]\tau_{min}, \tau_{max}[\text{ and } \widehat{\varepsilon}(\mathrm{U}) > \varepsilon_{min} \right\}$$
- $\widehat{\varepsilon} = \varepsilon - p_s \tau$ if stiffened gas EOS, $\widehat{\varepsilon} = \varepsilon$ otherwise

Positivity-preserving scheme

- Under which constraint, $\mathrm{U}_i^n \in G$ does imply $\mathrm{U}_i^{n+1} \in G$

1) Particular definition of the local acoustic impedances $\widetilde{z}_{qc}$

- $\widetilde{z}_{qc} = \rho_c^n \left(a_c^n + \sigma_v^{-1} \left| (\overline{\boldsymbol{u}}_q - \boldsymbol{u}_c^n) \cdot \boldsymbol{n}_{qc} \right| \right)$ Modified Dukowicz solver
- $\Delta t^n \leq \sigma_e \, \dfrac{m_c}{\sum_q \widetilde{z}_{qc}\, l_{qc}}$
- $\sigma_e \leq 1$
- $\sigma_v \leq \min\left(1 - \frac{\tau_{min}}{\tau_c^n}, \frac{\tau_{max}}{\tau_c^n} - 1, (1 - \frac{\varepsilon_{min}}{\varepsilon_c^n}) \left| \frac{\rho_c^n\, \varepsilon_c^n}{p_c^n} \right| \right)$

2) Additional constraint on the time step Δt^n

- $\Delta t^n \leq \sigma_e \, \dfrac{m_c}{\sum_q \widetilde{z}_{qc}\, l_{qc}}$ $\left(= \sigma_e \, \dfrac{|\omega_c|}{a_c^n \sum_q l_{qc}} \quad \text{if } \widetilde{z}_{qc} = \rho_c^n\, a_c^n \right)$
- $\sigma_e \leq 2$
- $\Delta t^n < \sigma_v \, \dfrac{|\omega_c^n|}{|\sum_q \overline{\boldsymbol{u}}_q \cdot l_{qc} \boldsymbol{n}_{qc}|}$ $\left(\iff \dfrac{\left| |\omega_c^{n+1}| - |\omega_c^n| \right|}{|\omega_c^n|} < \sigma_v \right)$
- $\sigma_v \leq \min\left(1 - \frac{\tau_{min}}{\tau_c^n}, \frac{\tau_{max}}{\tau_c^n} - 1, (1 - \frac{\varepsilon_{min}}{\varepsilon_c^n}) \left| \frac{\rho_c^n\, \varepsilon_c^n}{p_c^n} \right| \right)$

High-order polynomial extension

- $\mathrm{U}^n_{h,c}(\boldsymbol{x}) \in \mathbb{P}^K(\omega_c)$ piecewise polynomial reconstruction
- $\mathrm{U}^n_c = \dfrac{1}{m_c} \displaystyle\int_{\omega_c} \rho^n_{h,c}(\boldsymbol{x})\, \mathrm{U}^n_{h,c}(\boldsymbol{x})\, \mathrm{d}v$
- MUSCL, ENO, WENO, DG, ...

Generic scheme on the mass averaged values

- $\mathrm{U}^{n+1}_c = \mathrm{U}^n_c - \dfrac{\Delta t^n}{m_c} \displaystyle\sum_{q \in \mathcal{Q}_c} \overline{\mathrm{F}}_{qc} \,.\, l_{qc} \boldsymbol{n}_{qc}$
- $\overline{p}_{qc} = p_{qc} - \widetilde{z}_{qc}\, (\overline{\boldsymbol{u}}_q - \boldsymbol{u}_{qc}) \,.\, \boldsymbol{n}_{qc}$
- $\mathrm{U}_{qc} = \mathrm{U}^n_{h,c}(\boldsymbol{x}_q)$ and $p_{qc} = p(\mathrm{U}_{qc})$

X. Zhang and C.-W. Shu seminal work

- Positivity-preserving high-order schemes
- Decompose the high-order scheme in first-order-like schemes

High-order quadrature rule

- $\{(w_\alpha, y_\alpha)\}_{\alpha\in\Theta_c}$ are the positive quadrature weights and quadrature points, including the cell control point set, *i.e.* $\mathcal{Q}_c \subset \Theta_c$
- $\mathrm{U}_c^n = \frac{1}{m_c} \sum\limits_{\alpha\in\Theta_c} m_{\alpha c}\, \mathrm{U}_{\alpha c}$
- $m_{\alpha c} = w_\alpha\, m_c$ in the GLACE and EUCCLHYD schemes
- $m_{\alpha c} = w_\alpha\, \rho^0(\boldsymbol{X}_\alpha)\, |\Omega_c|$ in the CCDG scheme

U_c^n convex decomposition

- $\mathrm{U}_c^n = \dfrac{1}{m_c} \sum\limits_{\alpha\in\Theta_c\setminus\mathcal{Q}_c} m_{\alpha c}\, \mathrm{U}_{\alpha c} + \dfrac{1}{m_c} \sum\limits_{q\in\mathcal{Q}_c} m_{qc}\, \mathrm{U}_{qc} = \dfrac{m_c^\star}{m_c}\, \mathrm{U}_c^\star + \dfrac{1}{m_c} \sum\limits_{q\in\mathcal{Q}_c} m_{qc}\, \mathrm{U}_{qc}$
- $m_c^\star = \sum\limits_{\alpha\in\Theta_c\setminus\mathcal{Q}_c} m_{\alpha c}$ and $\mathrm{U}_c^\star = \dfrac{1}{m_c^\star} \sum\limits_{\alpha\in\Theta_c\setminus\mathcal{Q}_c} m_{\alpha c}\, \mathrm{U}_{\alpha c}$

Fundamental relation

- $\sum_{q \in \mathcal{Q}_c} l_{qc} \boldsymbol{n}_{qc} = \boldsymbol{0}$ The normals sum to zero

Artificial flux

- $\mathfrak{F}_{qc} = (-\mathfrak{u}_c,\ \mathbb{1}(1)\,\mathfrak{p}_{qc},\ \mathbb{1}(2)\,\mathfrak{p}_{qc},\ \mathfrak{p}_{qc}\,\mathfrak{u}_c)^{\mathrm{t}}$
- $\sum_{r \in \mathcal{Q}_c} \mathfrak{F}_{rc} \cdot l_{rc} \boldsymbol{n}_{rc} = 0$
- $\sum_{r \in \mathcal{Q}_c \backslash q} \mathfrak{F}_{rc} \cdot l_{rc} \boldsymbol{n}_{rc} = -\mathfrak{F}_{qc} \cdot l_{qc} \boldsymbol{n}_{qc}$

U_c^{n+1} convex decomposition

- $\mathsf{U}_c^{n+1} = \mathsf{U}_c^n - \frac{\Delta t^n}{m_c} \sum_{q \in \mathcal{Q}_c} \overline{\mathsf{F}}_{qc} \cdot l_{qc} \boldsymbol{n}_{qc} + \frac{\Delta t^n}{m_c} \sum_{q \in \mathcal{Q}_c} \mathfrak{F}_{qc} \cdot l_{qc} \boldsymbol{n}_{qc}$
- $\mathsf{U}_c^{n+1} = \frac{m_c^\star}{m_c} \mathsf{U}_c^\star + \sum_{q \in \mathcal{Q}_c} \frac{m_{qc}}{m_c} \underbrace{\left[\mathsf{U}_{qc} - \frac{\Delta t^n}{m_{qc}} \left(\overline{\mathsf{F}}_{qc} - \mathfrak{F}_{qc} \right) \cdot l_{qc} \boldsymbol{n}_{qc} \right]}_{\mathsf{V}_{qc}}$

New artificial flux

- $\overline{\mathfrak{F}}_r^q = (-\overline{\mathfrak{u}}_r^q,\ \mathbb{1}(1)\,\overline{\mathfrak{p}}_r^q,\ \mathbb{1}(2)\,\overline{\mathfrak{p}}_r^q,\ \overline{\mathfrak{p}}_r^q\,\overline{\mathfrak{u}}_r^q)^t$
- $\forall r \in \mathcal{Q}_c, \quad \overline{\mathfrak{F}}_r^q = \begin{cases} \overline{\mathrm{F}}_{qc}, & \text{if } r = q, \\ \mathfrak{F}_{rc}, & \text{otherwise,} \end{cases}$

U_c^{n+1} convex decomposition

- $\mathrm{U}_c^{n+1} = \dfrac{m_c^\star}{m_c}\,\mathrm{U}_c^\star + \displaystyle\sum_{q\in\mathcal{Q}_c} \dfrac{m_{qc}}{m_c}\,\mathrm{V}_{qc}$
- $\mathrm{V}_{qc} = \mathrm{U}_{qc} - \dfrac{\Delta t^n}{m_{qc}} \displaystyle\sum_{r\in\mathcal{Q}_c} \overline{\mathfrak{F}}_r^q \,.\, l_{rc}\boldsymbol{n}_{rc}$

First-order scheme

- $\mathrm{U}_c^{n+1} = \mathrm{U}_c^n - \dfrac{\Delta t^n}{m_c} \displaystyle\sum_{r\in\mathcal{Q}_c} \overline{\mathrm{F}}_{rc} \,.\, l_{rc}\boldsymbol{n}_{rc}$
- $\overline{\mathrm{F}}_{qc} = (-\overline{\boldsymbol{u}}_q,\ \mathbb{1}(1)\,\overline{p}_{qc},\ \mathbb{1}(2)\,\overline{p}_{qc},\ \overline{p}_{qc}\,\overline{\boldsymbol{u}}_q)^t$
- $\overline{p}_{rc} = p_c^n - \widetilde{z}_{rc}\,(\overline{\boldsymbol{u}}_r - \boldsymbol{u}_c^n) \,.\, \boldsymbol{n}_{rc}$

Conditions to mimic the first-order

- $$\sum_{r\in\mathcal{Q}_c} \overline{\mathfrak{p}}_r^q \, l_{rc}\boldsymbol{n}_{rc} = \sum_{r\in\mathcal{Q}_c} \left(p_{qc} - \widetilde{z}_{rc}^q \, (\overline{\mathfrak{u}}_r^q - \boldsymbol{u}_{qc}) \,\boldsymbol{.}\, \boldsymbol{n}_{rc} \right) l_{rc}\boldsymbol{n}_{rc}$$
- $$(\overline{p}_{qc} - \mathfrak{p}_{qc}) \, l_{qc}\boldsymbol{n}_{qc} = -\sum_{r\in\mathcal{Q}_c} \underbrace{\widetilde{z}_{rc}^q \, l_{rc} \, (\boldsymbol{n}_{rc} \otimes \boldsymbol{n}_{rc})}_{\mathsf{M}_{rc}^q} (\overline{\mathfrak{u}}_r^q - \boldsymbol{u}_{qc})$$
- $$\overline{p}_{qc} = p_{qc} - \widetilde{z}_{qc} \, (\overline{\boldsymbol{u}}_q - \boldsymbol{u}_{qc}) \,\boldsymbol{.}\, \boldsymbol{n}_{qc}$$

Artificial pressure

- $$\mathfrak{p}_{qc} \, l_{qc}\boldsymbol{n}_{qc} = p_{qc} \, l_{qc}\boldsymbol{n}_{qc} + \mathsf{M}_c^q \, (\mathfrak{u}_c - \boldsymbol{u}_{qc})$$

Artificial velocity

- $$\sum_{q\in\mathcal{Q}_c} \mathfrak{p}_{qc} \, l_{qc}\boldsymbol{n}_{qc} = 0$$
- $$\mathfrak{u}_c = \left(\sum_{q\in\mathcal{Q}_c} \mathsf{M}_c^q \right)^{-1} \sum_{q\in\mathcal{Q}_c} \left[\mathsf{M}_c^q \, \boldsymbol{u}_{qc} - p_{qc} \, l_{qc}\boldsymbol{n}_{qc} \right]$$

1) Particular definition of the local acoustic impedances $\widetilde{z}_{qc}$

- $\forall q \in \mathcal{Q}_c,\ \mathrm{U}_{qc} \in G$ and $\mathrm{U}_c^\star \in G$ Specific limitation procedure
- $\widetilde{z}_{rc}^q = \rho_{qc}\left(a_{qc} + \widetilde{\Gamma}\left|(\overline{\mathrm{u}}_r^q - \boldsymbol{u}_{qc})\,.\,\boldsymbol{n}_{rc}\right|\right)$
- $\Delta t \leq \sigma_e \dfrac{m_{qc}}{\sum_r \widetilde{z}_{rc}^q\, l_{rc}},\quad \forall q \in \mathcal{Q}_c$ with $\sigma_e \leq 1$
- $\sigma_v \leq \min\left(1 - \frac{\tau_{min}}{\tau_{qc}}, \frac{\tau_{max}}{\tau_{qc}} - 1, \left(1 - \frac{\varepsilon_{min}}{\varepsilon_{qc}}\right)\left|\frac{\varepsilon_{qc}}{\tau_{qc}\, p_{qc}}\right|\right)$

2) Additional constraint on the time step Δt^n

- $\forall q \in \mathcal{Q}_c,\ \mathrm{U}_{qc} \in G$ and $\mathrm{U}_c^\star \in G$ Specific limitation procedure
- $\Delta t \leq \sigma_e \dfrac{m_{qc}}{\sum_r \widetilde{z}_{rc}^q\, l_{rc}},\quad \forall q \in \mathcal{Q}_c$ with $\sigma_e \leq 2$
- $\Delta t \leq \sigma_v \dfrac{\tau_{qc}\, m_{qc}}{\left|\sum_{r \in \mathcal{Q}_c} \overline{\mathrm{u}}_r^q\,.\, l_{rc}\boldsymbol{n}_{rc}\right|}$
- $\sigma_v \leq \min\left(1 - \frac{\tau_{min}}{\tau_{qc}}, \frac{\tau_{max}}{\tau_{qc}} - 1, \left(1 - \frac{\varepsilon_{min}}{\varepsilon_{qc}}\right)\left|\frac{\varepsilon_{qc}}{\tau_{qc}\, p_{qc}}\right|\right)$

Mean value conservative limitation

- $\widetilde{\mathrm{U}_{h,c}^n}(\boldsymbol{x}) = \mathrm{U}_c^n + \theta\,(\mathrm{U}_{h,c}^n - \mathrm{U}_c^n)$
- $\theta \in [0,1]$ is the limiting coefficient to be determined

Requirements

- $\forall q \in \mathcal{Q}_c, \quad \widetilde{\mathrm{U}}_{qc} \equiv \widetilde{\mathrm{U}_{h,c}^n}(\boldsymbol{x}_{qc}) \in G$
- $\widetilde{\mathrm{U}}_c^\star \equiv \dfrac{1}{m_c^\star} \displaystyle\sum_{\alpha \in \Theta_c \setminus \mathcal{Q}_c} m_{\alpha c}\, \widetilde{\mathrm{U}_{h,c}^n}(\boldsymbol{x}_\alpha) \in G$

Specific volume limitation $\quad \tau \in [\tau_{min}, \tau_{max}]$

- $\widetilde{\tau_{h,c}^n}(\boldsymbol{x}) = \tau_c^n + \theta_\tau\,(\tau_{h,c}^n - \tau_c^n)$
- $\theta_\tau = \min(\theta_\tau^{min}, \theta_\tau^{max})$
- $\theta_\tau^{min} = \dfrac{\tau_c^n - \tau_{min}}{\tau_c^n - \tau_m^{min}}$ with $\tau_m^{min} = \min(\tau_c^\star, \min_{q \in \mathcal{Q}_c} \tau_{qc})$
- $\theta_\tau^{max} = \dfrac{\tau_{max} - \tau_c^n}{\tau_m^{max} - \tau_c^n}$ with $\tau_m^{max} = \max(\tau_c^\star, \max_{q \in \mathcal{Q}_c} \tau_{qc})$

Velocity and total energy limitation

- $\widetilde{\boldsymbol{u}_{h,c}^n}(\boldsymbol{x}) = \boldsymbol{u}_c^n + \theta_\varepsilon\,(\boldsymbol{u}_{h,c}^n - \boldsymbol{u}_c^n)$
- $\widetilde{e_{h,c}^n}(\boldsymbol{x}) = e_c^n + \theta_\varepsilon\,(e_{h,c}^n - e_c^n)$

Internal energy condition $\widehat{\varepsilon} > \varepsilon_{min}$

- $\varepsilon = e - \frac{1}{2}(\boldsymbol{u})^2$
- $\widehat{\varepsilon} = \varepsilon - p_s\,\tau$ if stiffened gas EOS, $\widehat{\varepsilon} = \varepsilon$ otherwise
- $\widetilde{\widehat{\varepsilon}_{h,c}^n}(x) = \widehat{\varepsilon}_c^n + \theta_\varepsilon\,(\widehat{\varepsilon}_{h,c}^n(x) - \widehat{\varepsilon}_c^n) + \dfrac{\theta_\varepsilon(1-\theta_\varepsilon)}{2}\,(\boldsymbol{u}_{h,c}^n(x) - \boldsymbol{u}_c^n)^2$
- θ_ε is chosen in optimal manner by solving this quadratic equation

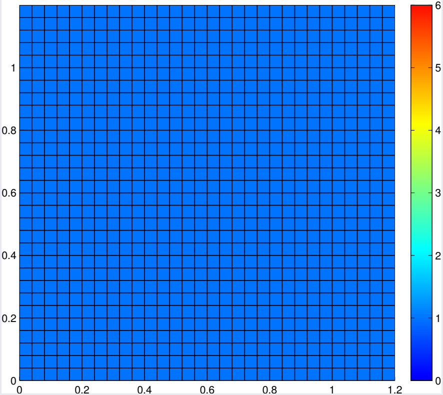
SSP Runge-Kutta method

- Convex combination of first-order forward Euler schemes
- We know there is a time step small enough ensuring the global high-order scheme to be positive

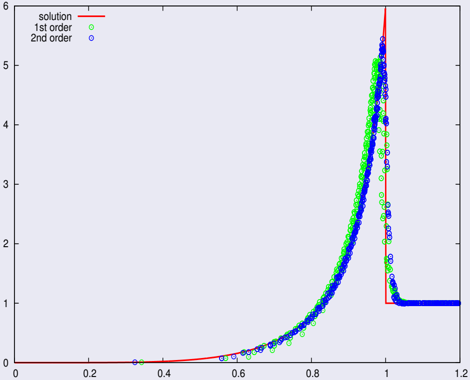
Practical applications - Iterative process

- At each time level n, we start from an initial time step Δt^n
- If at any Runge-Kutta stage the average of the numerical solution does not belonged to the admissible set then we return to time level n and take $\Delta t^n/2$ as new time step
- In the light of the theory previously developed, we know for sure this iterative process admits a limit

Sedov point blast problem on a Cartesian grid



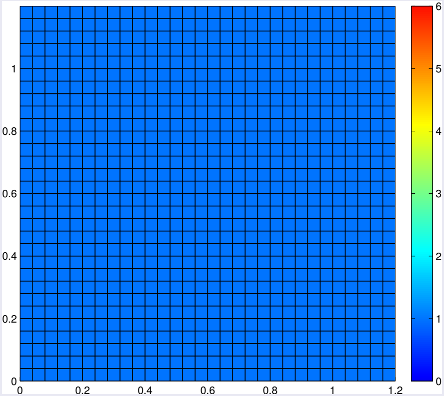
(e) Second-order scheme.



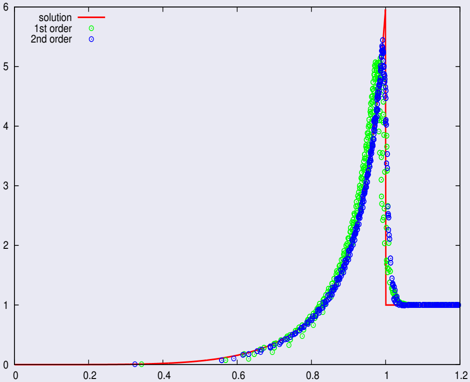
(f) Density profiles.

Fig: Point blast Sedov problem on a Cartesian grid made of 30 $\times$ 30 cells: density.

Sedov point blast problem on a Cartesian grid



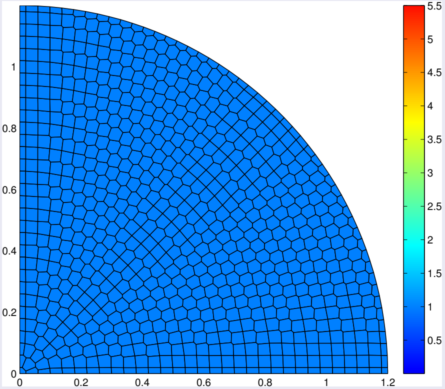
(e) Second-order scheme.



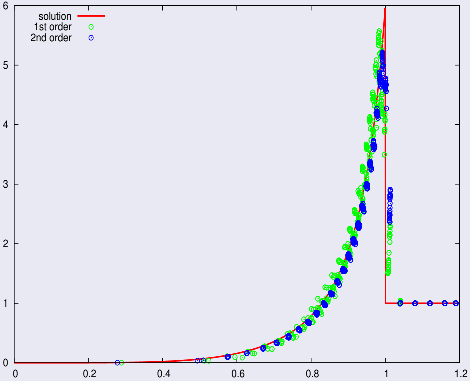
(f) Density profiles.

Fig: Point blast Sedov problem on a Cartesian grid made of 30 $\times$ 30 cells: density.

Sedov point blast problem on a polygonal grid



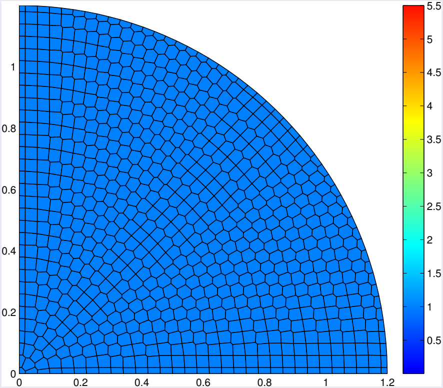
(g) Second-order scheme.



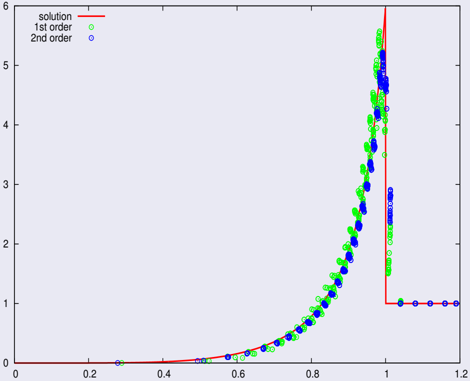
(h) Density profiles.

Fig: Point blast Sedov problem on a mesh made of 775 polygonal cells: density.

Sedov point blast problem on a polygonal grid



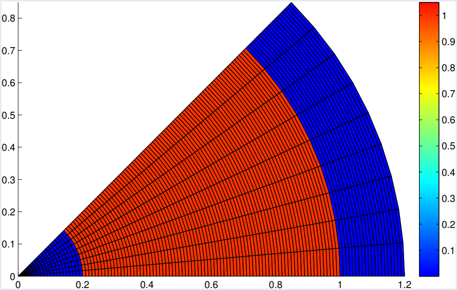
(g) Second-order scheme.



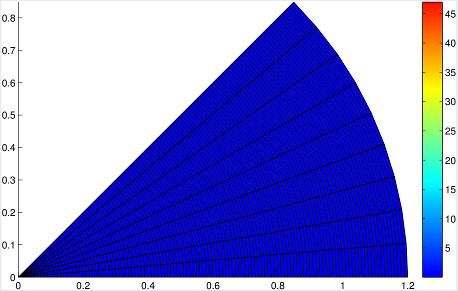
(h) Density profiles.

Fig: Point blast Sedov problem on a mesh made of 775 polygonal cells: density.

Air-water-air problem on a polar grid



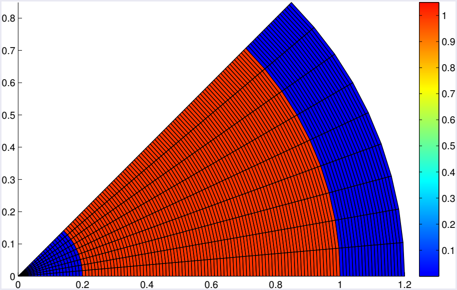
(i) Density map.



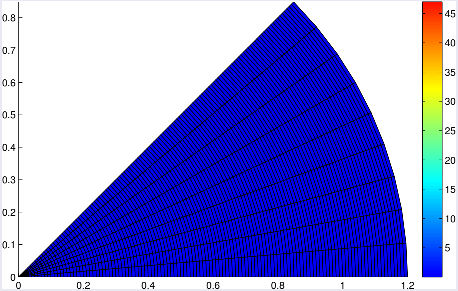
(j) Kinetic energy map.

Fig: Air-water-air problem on a polar grid made of 120×9 cells with second-order scheme.

Air-water-air problem on a polar grid



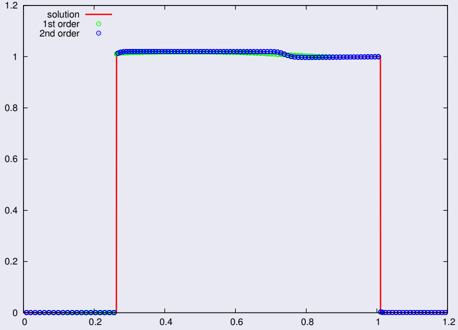
(i) Density map.



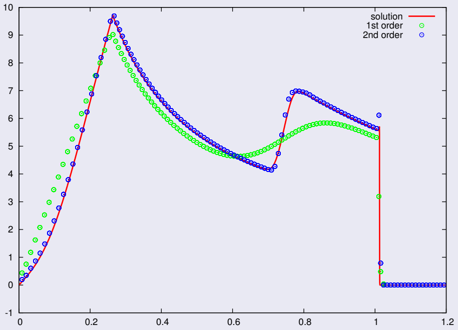
(j) Kinetic energy map.

Fig: Air-water-air problem on a polar grid made of 120×9 cells with second-order scheme.

Air-water-air problem on a polar grid



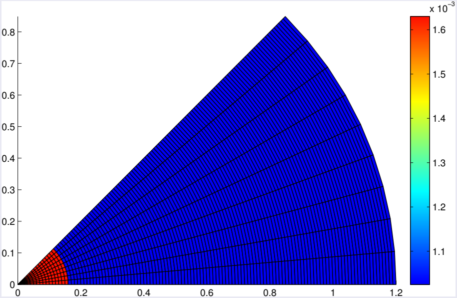
(k) Density profiles.



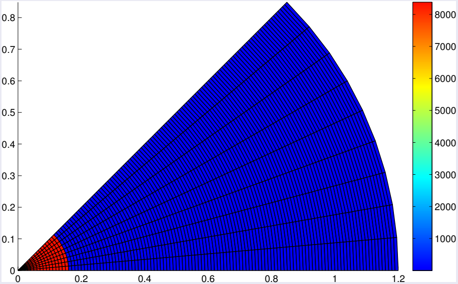
(l) Velocity profiles.

Fig: Air-water-air problem on a polar grid made of 120 $\times$ 9 cells.

Underwater TNT explosion problem on a polar grid



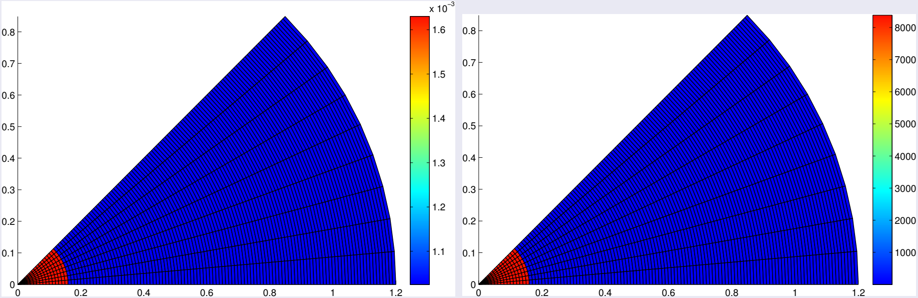
(m) Density map.



(n) Pressure map.

Fig: Underwater TNT explosion problem on a polar grid made of 120×9 cells with second-order scheme.

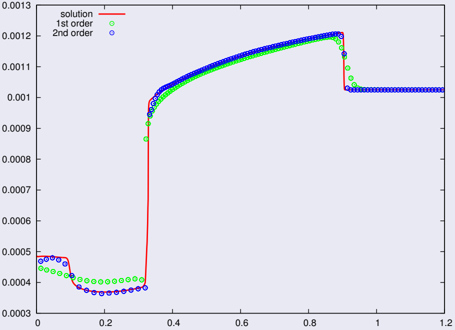
Underwater TNT explosion problem on a polar grid



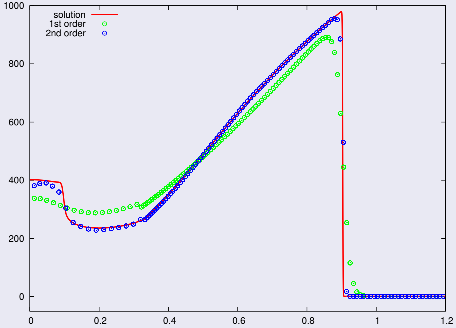
(m) Density map. (n) Pressure map.

Fig: Underwater TNT explosion problem on a polar grid made of 120 $\times$ 9 cells with second-order scheme.

Underwater TNT explosion problem on a polar grid



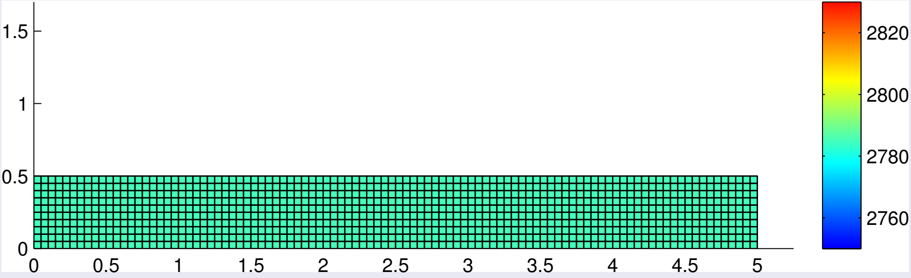
(o) Density profiles.



(p) Pressure profiles.

Fig: Underwater TNT explosion problem on a polar grid made of 120×9 cells.

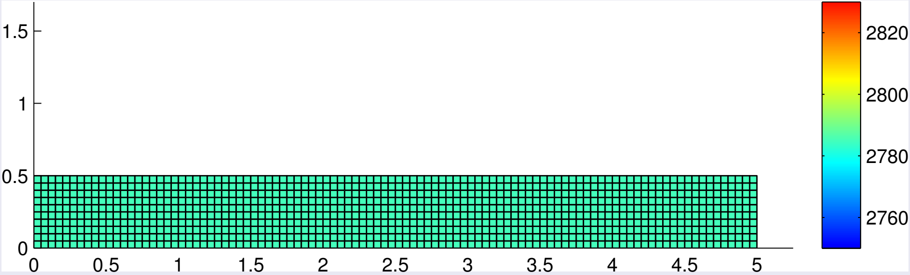
Projectile impact problem on a Cartesian grid



(q) Density maps.

Fig: Projectile impact problem on a Cartesian grid made of 100×10 cells with second-order scheme.

Projectile impact problem on a Cartesian grid



(q) Density maps.

Fig: Projectile impact problem on a Cartesian grid made of 100 $\times$ 10 cells with second-order scheme.

Conclusions

- Demonstration of positivity of one-dimensional Lagrangian schemes
- For both first-order scheme and high-order schemes
- For both ideal and non-ideal equations of state
- Two different techniques used
  - Particular definition of the local acoustic impedances approximation
  - Additional constraint of the time step
- Extension to the two-dimensional case
- Theory fits a wide number of existing cell-centered Lagrangian schemes
- Improvement of the robustness

Perspectives

- High-order limitation on moving high-order geometries
- Extension to ALE
- Extension to magneto-hydrodynamics (FCM)
- Extension to 3D

Articles published on this topic

F. VILAR, P.-H. MAIRE and C.-W. SHU, *Positivity-preserving cell-centered Lagrangian schemes for multi-material compressible flows: From first-order to high-orders.* Journal of Computational Physics, 2015. **To be submitted in the next days**.

F. VILAR, P.-H. MAIRE and R. ABGRALL, *A discontinuous Galerkin discretization for solving the two-dimensional gas dynamics equations written under total Lagrangian formulation on general unstructured grids.* Journal of Computational Physics, 2014.

F. VILAR, *A discontinuous Galerkin discretization for solving the two-dimensional gas dynamics equations written under total Lagrangian formulation on general unstructured grids.* Computers and Fluids, 2012.

F. VILAR, P.-H. MAIRE and R. ABGRALL, *Cell-centered discontinuous Galerkin discretizations for two-dimensional scalar conservation laws on unstructured grids and for one-dimensional Lagrangian hydrodynamics.* Computers and Fluids, 2010.