

High-order cell-centered DG scheme for Lagrangian hydrodynamics

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energie atomique • energies alternatives



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- 2 Cell-Centered Lagrangian schemes
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- 5 Numerical results
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DG schemes

- Natural extension of Finite Volume method
- Piecewise polynomial approximation of the solution in the cells
- High-order scheme to achieve high accuracy

Procedure

- Local variational formulation
- Choice of the numerical fluxes (global L^2 stability, entropy inequality)
- Time discretization - TVD multistep Runge-Kutta



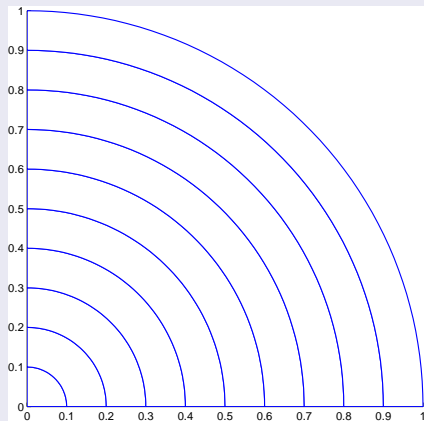
C.-W. SHU, *Discontinuous Galerkin methods: General approach and stability*. 2008.

- Limitation - vertex-based hierarchical slope limiters

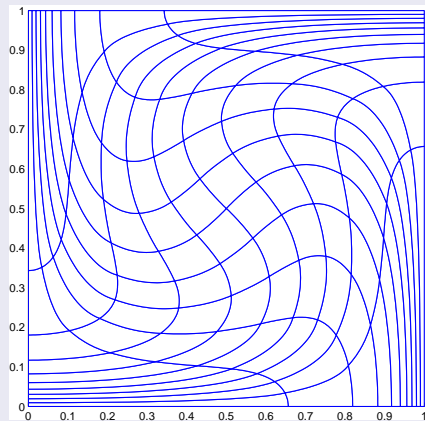


D. KUZMIN, *A vertex-based hierarchical slope limiter for p-adaptive discontinuous Galerkin methods*. J. Comp. Appl. Math., 2009.

Circular polar grid: 10×1 cells



Taylor-Green exact motion



V. DOBREV, T. ELLIS, T. KOLEV AND R. RIEBEN, *High Order Curvilinear Finite Elements for Lagrangian Hydrodynamics. Part I: General Framework*, 2010. Presentation available at <https://computation.llnl.gov/casc/blast/blast.html>

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Finite volume schemes on moving mesh

- J. K. Dukowicz: CAVEAT scheme
A computer code for fluid dynamics problems with large distortion and internal slip, 1986
- B. Després: GLACE scheme
Lagrangian Gas Dynamics in Two Dimensions and Lagrangian systems, 2005
- P.-H. Maire: EUCCLHYD scheme
A cell-centered Lagrangian scheme for two-dimensional compressible flow problems, 2007
- G. Kluth: Hyperelasticity
Discretization of hyperelasticity with a cell-centered Lagrangian scheme, 2010
- S. Del Pino: Curvilinear Finite Volume method
A curvilinear finite-volume method to solve compressible gas dynamics in semi-Lagrangian coordinates, 2010
- P. Hoch: Finite Volume method on unstructured conical meshes
Extension of ALE methodology to unstructured conical meshes, 2011

DG scheme on initial mesh

- R. Loubère: DG scheme for Lagrangian hydrodynamics
A Lagrangian Discontinuous Galerkin-type method on unstructured meshes to solve hydrodynamics problems, 2004

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Flow transformation of the fluid

- The fluid flow is described mathematically by the continuous transformation, Φ , so-called mapping such as $\Phi : \mathbf{X} \longrightarrow \mathbf{x} = \Phi(\mathbf{X}, t)$

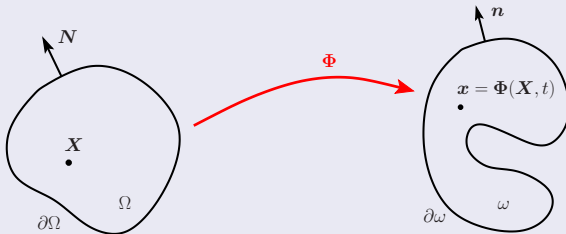


Figure: Notation for the flow map.

where \mathbf{X} is the Lagrangian (initial) coordinate, \mathbf{x} the Eulerian (actual) coordinate, \mathbf{N} the Lagrangian normal and \mathbf{n} the Eulerian normal

Deformation Jacobian matrix: deformation gradient tensor

- $\mathbf{F} = \nabla_{\mathbf{X}}\Phi = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$ and $J = \det \mathbf{F} > 0$

Trajectory equation

- $\frac{d\mathbf{x}}{dt} = \mathbf{U}(\mathbf{x}, t), \quad \mathbf{x}(\mathbf{X}, 0) = \mathbf{X}$

Material time derivative

- $\frac{d}{dt}f(\mathbf{x}, t) = \frac{\partial}{\partial t}f(\mathbf{x}, t) + \mathbf{U} \cdot \nabla_{\mathbf{x}}f(\mathbf{x}, t)$

Transformation formulas

- $F d\mathbf{X} = d\mathbf{x}$ Change of shape of infinitesimal vectors
- $\rho^0 = \rho J$ Mass conservation
- $J dV = dv$ Measure of the volume change
- $JF^{-t} \mathbf{N} dS = \mathbf{n} ds$ **Nanson formula**

Differential operators transformations

- $\nabla_{\mathbf{x}} P = \frac{1}{J} \nabla_{\mathbf{X}} \cdot (P J F^{-t})$ Gradient operator
- $\nabla_{\mathbf{x}} \cdot \mathbf{U} = \frac{1}{J} \nabla_{\mathbf{X}} \cdot (J F^{-1} \mathbf{U})$ Divergence operator

Piola compatibility condition

- $\nabla_x \cdot \mathbf{G} = \mathbf{0}$, where $\mathbf{G} = \mathbf{J}\mathbf{F}^{-t}$ is the cofactor matrix of \mathbf{F}

$$\int_{\Omega} \nabla_x \cdot \mathbf{G} dV = \int_{\partial\Omega} \mathbf{G} \mathbf{N} dS = \int_{\partial\omega} \mathbf{n} ds = \mathbf{0}$$

Gas dynamics system written in its total Lagrangian form

- $\frac{d\mathbf{F}}{dt} - \nabla_x \mathbf{U} = \mathbf{0}$ Deformation gradient tensor equation
- $\rho^0 \frac{d}{dt} \left(\frac{1}{\rho} \right) - \nabla_x \cdot (\mathbf{G}^t \mathbf{U}) = \mathbf{0}$ Specific volume equation
- $\rho^0 \frac{d\mathbf{U}}{dt} + \nabla_x \cdot (P \mathbf{G}) = \mathbf{0}$ Momentum equation
- $\rho^0 \frac{dE}{dt} + \nabla_x \cdot (\mathbf{G}^t P \mathbf{U}) = \mathbf{0}$ Total energy equation

Thermodynamical closure

- EOS: $P = P(\rho, \varepsilon)$ where $\varepsilon = E - \frac{1}{2} \mathbf{U}^2$

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$(s + 1)^{\text{th}}$ order DG discretization

- Let $\{\Omega_c\}_c$ be a partition of the domain Ω into polygonal cells
- $\{\sigma_k^c\}_{k=0\dots K}$ basis of $\mathbb{P}^s(\Omega_c)$, where $K + 1 = \frac{(s+1)(s+2)}{2}$
- $\phi_h^c(\mathbf{X}, t) = \sum_{k=0}^K \phi_k^c(t) \sigma_k^c(\mathbf{X})$ approximate function of $\phi(\mathbf{X}, t)$ on Ω_c

Definitions

- Center of mass $\mathcal{X}_c = (\mathcal{X}_c, \mathcal{Y}_c)^t = \frac{1}{m_c} \int_{\Omega_c} \rho^0(\mathbf{X}) \mathbf{X} dV$,
 where m_c is the constant mass of the cell Ω_c
- The mean value $\langle \phi \rangle_c = \frac{1}{m_c} \int_{\Omega_c} \rho^0(\mathbf{X}) \phi(\mathbf{X}) dV$
 of the function ϕ over the cell Ω_c
- The associated scalar product $(\phi \cdot \psi)_c = \int_{\Omega_c} \rho^0(\mathbf{X}) \phi(\mathbf{X}) \psi(\mathbf{X}) dV$

Taylor expansion on the cell, located at the center of mass

$$\phi(\mathbf{X}) = \phi(\mathbf{x}_c) + \sum_{k=1}^s \sum_{j=0}^k \frac{(\mathbf{X} - \mathbf{x}_c)^{k-j} (\mathbf{Y} - \mathbf{y}_c)^j}{j!(k-j)!} \frac{\partial^k \phi}{\partial \mathbf{X}^{k-j} \partial \mathbf{Y}^j}(\mathbf{x}_c) + o(\|\mathbf{X} - \mathbf{x}_c\|^s)$$

$(s + 1)^{\text{th}}$ order scheme polynomial Taylor basis

- The first-order polynomial component and the associated basis function

$$\phi_0^c = \langle \phi \rangle_c \quad \text{and} \quad \sigma_0^c = 1$$

- The k^{th} -order polynomial components and the associated basis functions

$$\phi_{\frac{k(k+1)}{2}+j}^c = (\Delta X_c)^{k-j} (\Delta Y_c)^j \frac{\partial^k \phi}{\partial X^{k-j} \partial Y^j}(\mathbf{x}_c),$$

$$\sigma_{\frac{k(k+1)}{2}+j}^c = \frac{1}{j!(k-j)!} \left[\left(\frac{X - x_c}{\Delta X_c} \right)^{k-j} \left(\frac{Y - y_c}{\Delta Y_c} \right)^j - \left\langle \left(\frac{X - x_c}{\Delta X_c} \right)^{k-j} \left(\frac{Y - y_c}{\Delta Y_c} \right)^j \right\rangle_c \right],$$

where $0 < k \leq s$, $j = 0 \dots k$, $\Delta X_c = \frac{X_{\max} - X_{\min}}{2}$ and $\Delta Y_c = \frac{Y_{\max} - Y_{\min}}{2}$



H. LUO, J. D. BAUM AND R. LÖHNER, *A DG method based on a Taylor basis for the compressible flows on arbitrary grids*. J. Comp. Phys., 2008.

Outcome

- First moment associated to the basis function $\sigma_0^c = 1$ is the mass averaged value

$$\phi_0^c = \langle \phi \rangle_c$$

- The successive moments can be identified as the successive derivatives of the function expressed at the center of mass of the cell

$$\phi_{\frac{k(k+1)}{2}+j}^c = (\Delta X_c)^{k-j} (\Delta Y_c)^j \frac{\partial^k \phi}{\partial X^{k-j} \partial Y^j}(\mathbf{x}_c)$$

- The first basis function is orthogonal to the other ones

$$(\sigma_0^c \cdot \sigma_k^c)_c = m_c \delta_{0k}$$

- **Same basis functions regardless the shape of the cells** (squares, triangles, generic polygonal cells)

Lagrangian gas dynamics equation type

- $\rho^0 \frac{d\phi}{dt} + \nabla_X \cdot (\mathbf{G}^t \mathbf{f}) = 0$, where \mathbf{f} is the flux function
 $\mathbf{G} = \mathbf{J}\mathbf{F}^{-t}$ is the cofactor matrix of \mathbf{F}

Local variational formulations

- $$\int_{\Omega_c} \rho^0 \frac{d\phi}{dt} \sigma_j^c dV = \sum_{k=0}^K \frac{d\phi_k^c}{dt} \int_{\Omega_c} \rho^0 \sigma_j^c \sigma_k^c dV$$

$$= \int_{\Omega_c} \mathbf{f} \cdot \mathbf{G} \nabla_X \sigma_j^c dV - \int_{\partial\Omega_c} \bar{\mathbf{f}} \cdot \sigma_j^c \mathbf{G} \mathbf{N} dS$$

Geometric Conservation Law (GCL)

- Equation on the first moment of the specific volume

$$\int_{\Omega_c} \frac{dJ}{dt} dV = \frac{d|\omega_c|}{dt} = \int_{\Omega_c} \nabla_X \cdot (\mathbf{G}^t \mathbf{U}) dV = \int_{\partial\Omega_c} \bar{\mathbf{U}} \cdot \mathbf{G} \mathbf{N} dS$$

Mass matrix properties

- $\int_{\Omega_c} \rho^0 \sigma_j^c \sigma_k^c dV = (\sigma_j^c \cdot \sigma_k^c)_c$ generic coefficient of the symmetric positive definite mass matrix
- $(\sigma_0^c \cdot \sigma_k^c)_c = m_c \delta_{0k}$ mass averaged equation is independent of the other polynomial basis components equations

Interior terms

- $\int_{\Omega_c} \mathbf{f} \cdot \mathbf{G} \nabla_X \sigma_j^c dV$ is evaluated through the use of a two-dimensional high-order quadrature rule

Boundary terms

- $\int_{\partial\Omega_c} \bar{\mathbf{f}} \cdot \sigma_j^c \mathbf{G} \mathbf{N} dS$ required a specific treatment to ensure the GCL
- It remains to determine the numerical fluxes

Requirements

- **Consistency** of vector $\mathbf{GN}dS = \mathbf{n}ds$ at the interfaces of the cells
- **Continuity** of vector \mathbf{GN} at cell interfaces on both sides of the interface
- **Preservation of uniform flows**, $\mathbf{G} = \mathbf{JF}^{-t}$ the cofactor matrix

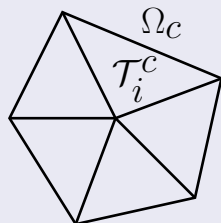
$$\int_{\Omega_c} \mathbf{G} \nabla_X \sigma_j^c dV = \int_{\partial\Omega_c} \sigma_j^c \mathbf{GN}dS \iff \int_{\Omega_c} \sigma_j^c (\nabla_X \cdot \mathbf{G}) dV = \mathbf{0}$$

Generalization of the weak form of the Piola compatibility condition

Tensor F discretization

- Discretization of tensor \mathbf{F} by means of a mapping defined on triangular cells
- Partition of the polygonal cells in the initial configuration into non-overlapping triangles

$$\Omega_c = \bigcup_{i=1}^{ntri} \mathcal{T}_i^c$$



$(s + 1)^{\text{th}}$ order continuous mapping function

- We develop Φ on the Finite Elements basis functions Λ_q^i in \mathcal{T}_i of degree s

$$\Phi_h^i(\mathbf{X}, t) = \sum_{q \in \mathcal{Q}(i)} \Lambda_q^i(\mathbf{X}) \Phi_q(t),$$

where $\mathcal{Q}(i)$ is the \mathcal{T}_i control points set, including the vertices $\{p^-, p, p^+\}$

- $\Phi_q(t) = \Phi(\mathbf{X}_q, t) = \mathbf{x}_q$
- $\frac{d\Phi_q}{dt} = \mathbf{U}_q \implies \frac{d}{dt} F_i(\mathbf{X}, t) = \sum_{q \in \mathcal{Q}(i)} \mathbf{U}_q(t) \otimes \nabla_{\mathbf{X}} \Lambda_q^i(\mathbf{X})$



G. KLUTH AND B. DESPRÉS, *Discretization of hyperelasticity on unstructured mesh with a cell-centered Lagrangian scheme*. J. Comp. Phys., 2010.

Outcome

- Satisfaction of the Piola compatibility condition **everywhere**
- **Consistency** and **continuity** of the Eulerian normal **GN**

Example of the fluid flow mapping in the fourth order case

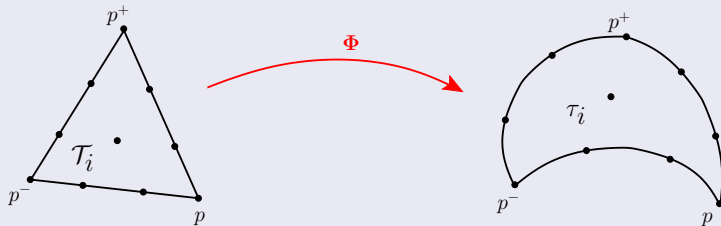


Figure: Nodes arrangement for a cubic Lagrange Finite Element mapping.

Curved edges definition using $s + 1$ control points

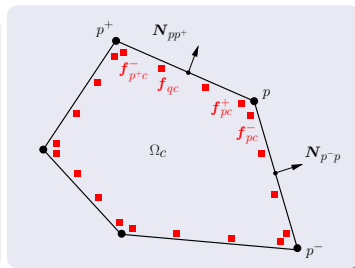
- Projection of the continuous mapping function Φ on the face f_{pp^+}

$$\mathbf{x}|_{pp^+}(\zeta) = \mathbf{x}_p \lambda_p(\zeta) + \sum_{q \in \mathcal{Q}(pp^+) \setminus \{p, p^+\}} \mathbf{x}_q \lambda_q(\zeta) + \mathbf{x}_{p^+} \lambda_{p^+}(\zeta),$$

where $\mathcal{Q}(pp^+)$ is the face control points set, $\zeta \in [0, 1]$ the curvilinear abscissa and λ_q the 1D Finite Element basis functions of degree s

Local variational formulations

$$\bullet \int_{\Omega_c} \rho^0 \frac{d\phi}{dt} \sigma_j^c dV = - \sum_{i=1}^{ntri} \int_{T_i^c} \mathbf{G} \nabla_X \sigma_j^c \cdot \mathbf{f} dV + \sum_{p \in \mathcal{P}(c)} \int_p^{p^+} \bar{\mathbf{f}} \cdot \sigma_j^c \mathbf{G} \mathbf{N} dL$$



Polynomial assumptions on face f_{pp^+}

$$\bullet \mathbf{f}|_{f_{pp^+}}(\zeta) = \mathbf{f}_{pc}^+ \lambda_p(\zeta) + \sum_{q \in \{p, p^+\}} \mathbf{f}_{qc} \lambda_q(\zeta) + \mathbf{f}_{p+c}^- \lambda_{p^+}(\zeta)$$

Polynomial properties on face f_{pp^+}

$$\bullet \mathbf{G} \mathbf{N} dL|_{f_{pp^+}}(\zeta) = \mathbf{n} dL|_{f_{pp^+}} = \frac{\partial \mathbf{x}}{\partial \zeta} d\zeta|_{f_{pp^+}} \times \mathbf{e}_z = \sum_q \frac{\partial \lambda_q}{\partial \zeta}(\zeta) (\mathbf{x}_q \times \mathbf{e}_z)$$

$$\bullet \sigma_j^c|_{f_{pp^+}}(\zeta) = \sigma_j^c(\mathbf{X}_p) \lambda_p(\zeta) + \sum_{q \in \{p, p^+\}} \sigma_j^c(\mathbf{X}_q) \lambda_q(\zeta) + \sigma_j^c(\mathbf{X}_{p^+}) \lambda_{p^+}(\zeta)$$

Fundamental assumptions

- $\mathbf{U}_{pc}^{\pm} = \mathbf{U}_p$, $\forall c \in \mathcal{C}(p)$ and $\mathbf{U}_{qL} = \mathbf{U}_{qR} = \mathbf{U}_q$
- $\overline{P\mathbf{U}} = \overline{P} \overline{\mathbf{U}} \implies (P\mathbf{U})_{pc}^{\pm} = P_{pc}^{\pm} \mathbf{U}_p$ and $(P\mathbf{U})_{qc} = P_{qc} \mathbf{U}_q$

Procedure

- Analytical integration + index permutation

Weighted control point normals

- $l_{pc}^{+,j} \mathbf{n}_{pc}^{+,j} = \left(\int_0^1 \lambda_{p|_{pp^+}}(\zeta) \sigma_{j|_{pp^+}}(\zeta) \frac{\partial \mathbf{x}}{\partial \zeta} d\zeta \Big|_{pp^+} \right) \times \mathbf{e}_z$
- $l_{pc}^{-,j} \mathbf{n}_{pc}^{-,j} = \left(\int_0^1 \lambda_{p|_{p-p}}(\zeta) \sigma_{j|_{p-p}}(\zeta) \frac{\partial \mathbf{x}}{\partial \zeta} d\zeta \Big|_{p-p} \right) \times \mathbf{e}_z$
- $l_{pc}^j \mathbf{n}_{pc}^j = l_{pc}^{-,j} \mathbf{n}_{pc}^{-,j} + l_{pc}^{+,j} \mathbf{n}_{pc}^{+,j}$
- $l_{qc}^j \mathbf{n}_{qc}^j = \left(\int_0^1 \lambda_{q|_{pp^+}}(\zeta) \sigma_{j|_{pp^+}}(\zeta) \frac{\partial \mathbf{x}}{\partial \zeta} d\zeta \Big|_{pp^+} \right) \times \mathbf{e}_z$

j^{th} moment of the subcell forces

- $\mathbf{F}_{pc}^j = P_{pc}^- l_{pc}^{-,j} \mathbf{n}_{pc}^{-,j} + P_{pc}^+ l_{pc}^{+,j} \mathbf{n}_{pc}^{+,j}$ and $\mathbf{F}_{qc}^j = P_{qc} l_{qc}^j \mathbf{n}_{qc}^j$

Semi-discrete equations GCL compatible

$$\int_{\Omega_c} \rho^0 \frac{d}{dt} \left(\frac{1}{\rho} \right) \sigma_j^c dV = - \sum_{i=1}^{ntri} \int_{\mathcal{T}_i^c} \mathbf{U} \cdot \mathbf{G} \nabla_X \sigma_j^c dV + \sum_{p \in \mathcal{P}(c)} \left(\mathbf{U}_p \cdot l_{pc}^j \mathbf{n}_{pc}^j + \sum_{q \in \{p, p^+\}} \mathbf{U}_q \cdot l_{qc}^j \mathbf{n}_{qc}^j \right)$$

$$\int_{\Omega_c} \rho^0 \frac{d\mathbf{U}}{dt} \sigma_j^c dV = \sum_{i=1}^{ntri} \int_{\mathcal{T}_i^c} P \mathbf{G} \nabla_X \sigma_j^c dV - \sum_{p \in \mathcal{P}(c)} \left(\mathbf{F}_{pc}^j + \sum_{q \in \{p, p^+\}} \mathbf{F}_{qc}^j \right)$$

$$\int_{\Omega_c} \rho^0 \frac{dE}{dt} \sigma_j^c dV = \sum_{i=1}^{ntri} \int_{\mathcal{T}_i^c} P \mathbf{U} \cdot \mathbf{G} \nabla_X \sigma_j^c dV - \sum_{p \in \mathcal{P}(c)} \left(\mathbf{U}_p \cdot \mathbf{F}_{pc}^j + \sum_{q \in \{p, p^+\}} \mathbf{U}_q \cdot \mathbf{F}_{qc}^j \right)$$

Equation on the first moment of the specific volume

$$\bullet \frac{d|\omega_c|}{dt} = \int_{\partial\Omega_c} \bar{\mathbf{U}} \cdot \mathbf{G} \mathbf{N} dL = \sum_{p \in \mathcal{P}(c)} \left(\mathbf{U}_p \cdot l_{pc}^0 \mathbf{n}_{pc}^0 + \sum_{q \in \mathcal{Q}(\{p, p^+\})} \mathbf{U}_q \cdot l_{qc}^0 \mathbf{n}_{qc}^0 \right)$$

Entropic semi-discrete equation

- Fundamental assumption $\overline{P\mathbf{U}} = \overline{P}\overline{\mathbf{U}}$
- The use of variational formulations and Piola condition leads to

$$\int_{\Omega_c} \rho^0 \theta \frac{d\eta}{dt} dV = \int_{\partial\Omega_c} (\overline{P} - P_h)(\mathbf{U}_h - \overline{\mathbf{U}}) \cdot \mathbf{GN} dS,$$

where η is the specific entropy and θ the absolute temperature defined by means of the Gibbs identity

Entropic semi-discrete equation

- A sufficient condition to satisfy $\int_{\Omega_c} \rho^0 \theta \frac{d\eta}{dt} dV \geq 0$ is

$$\overline{P} - P_h = -Z(\overline{\mathbf{U}} - \mathbf{U}_h) \cdot \frac{\mathbf{GN}}{\|\mathbf{GN}\|} = -Z(\overline{\mathbf{U}} - \mathbf{U}_h) \cdot \mathbf{n},$$

where $Z \geq 0$ has the physical dimension of a density times a velocity

Subcell forces definitions

- $$\mathbf{F}_{pc}^j = P_{pc}^- l_{pc}^{-,j} \mathbf{n}_{pc}^{-,j} + P_{pc}^+ l_{pc}^{+,j} \mathbf{n}_{pc}^{+,j} \quad \text{and} \quad \mathbf{F}_{qc}^j = P_{qc} l_{qc}^j \mathbf{n}_{qc}^j$$

j^{th} moment of the control point subcell forces

- The use of $\bar{P} = P_h^c - Z_c (\bar{\mathbf{U}} - \mathbf{U}_h^c) \cdot \mathbf{n}$ to calculate \mathbf{F}_{pc}^j and \mathbf{F}_{qc}^j leads to

$$\mathbf{F}_{pc}^j = P_h^c(\mathbf{X}_p, t) l_{pc}^j \mathbf{n}_{pc}^j - M_{pc}^j (\mathbf{U}_p - \mathbf{U}_h^c(\mathbf{X}_p, t)),$$

$$\mathbf{F}_{qc}^j = P_h^c(\mathbf{X}_q, t) l_{qc}^j \mathbf{n}_{qc}^j - M_{qc}^j (\mathbf{U}_q - \mathbf{U}_h^c(\mathbf{X}_q, t)),$$

$$M_{pc}^j = Z_c \left(l_{pc}^{-,j} \mathbf{n}_{pc}^{-,j} \otimes \mathbf{n}_{pc}^{-,0} + l_{pc}^{+,j} \mathbf{n}_{pc}^{+,j} \otimes \mathbf{n}_{pc}^{+,0} \right) \quad \text{and} \quad M_{qc}^j = Z_c l_{qc}^j \mathbf{n}_{qc}^j \otimes \mathbf{n}_{qc}^0$$

Momentum and total energy conservation

- $$\sum_{c \in \mathcal{C}(p)} \mathbf{F}_{pc}^0 = \mathbf{0} \quad \text{and} \quad \mathbf{F}_{qL}^0 + \mathbf{F}_{qR}^0 = \mathbf{0}$$

Nodal velocity

- $$M_p \mathbf{U}_p = \sum_{c \in \mathcal{C}(p)} [P_h^c(\mathbf{X}_p, t) l_{pc}^0 \mathbf{n}_{pc}^0 + M_{pc}^0 \mathbf{U}_h^c(\mathbf{X}_p, t)],$$

where $M_p = \sum_{c \in \mathcal{C}(p)} M_{pc}^0$ is a **positive definite** matrix

Face control point velocity

- $$M_q \mathbf{U}_q = M_q \left(\frac{Z_L \mathbf{U}_h^L(\mathbf{X}_q) + Z_R \mathbf{U}_h^R(\mathbf{X}_q)}{Z_L + Z_R} \right) - \frac{P_h^R(\mathbf{X}_q) - P_h^L(\mathbf{X}_q)}{Z_L + Z_R} l_{qL}^0 \mathbf{n}_{qL}^0,$$

where $M_q = \frac{1}{Z_R} M_{qR}^0 = \frac{1}{Z_L} M_{qL}^0 = l_{qL}^0 \mathbf{n}_{qL}^0 \otimes \mathbf{n}_{qL}^0$ is **positive semi-definite**

1D approximate Riemann problem solution

- $$(\mathbf{U}_q \cdot \mathbf{n}_{qL}^0) = \left(\frac{Z_L \mathbf{U}_h^L(\mathbf{X}_q) + Z_R \mathbf{U}_h^R(\mathbf{X}_q)}{Z_L + Z_R} \right) \cdot \mathbf{n}_{qL}^0 - \frac{P_h^R(\mathbf{X}_q) - P_h^L(\mathbf{X}_q)}{Z_L + Z_R}$$

Tangential component of the face control point velocity

$$\bullet (\mathbf{U}_q \cdot \mathbf{t}_{qL}^0) = \left(\frac{Z_L \mathbf{U}_h^L(\mathbf{X}_q) + Z_R \mathbf{U}_h^R(\mathbf{X}_q)}{Z_L + Z_R} \right) \cdot \mathbf{t}_{qL}^0$$

Face control point velocity

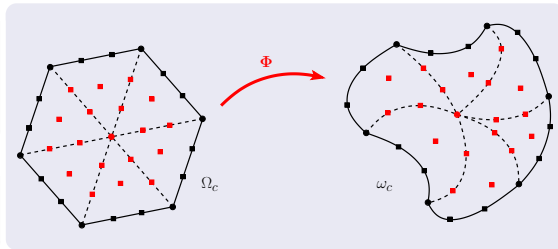
$$\bullet \mathbf{U}_q = \frac{Z_L \mathbf{U}_h^L(\mathbf{X}_q) + Z_R \mathbf{U}_h^R(\mathbf{X}_q)}{Z_L + Z_R} - \frac{P_h^R(\mathbf{X}_q) - P_h^L(\mathbf{X}_q)}{Z_L + Z_R} \mathbf{n}_{qL}^0$$

Deformation tensor

$$\bullet \frac{d}{dt} \mathbf{F}_i = \sum_{Q \in \mathcal{Q}(i)} \mathbf{U}_Q \otimes \nabla_{\mathbf{x}} \Lambda_Q^i$$

Interior points velocity

$$\bullet \mathbf{U}_Q = \mathbf{U}_h^c(\mathbf{X}_Q, t)$$



Riemann invariants differentials

- $d\alpha_t = d\mathbf{U} \cdot \mathbf{t}$
- $d\alpha_- = d\left(\frac{1}{\rho}\right) - \frac{1}{\rho a} d\mathbf{U} \cdot \mathbf{n}$
- $d\alpha_+ = d\left(\frac{1}{\rho}\right) + \frac{1}{\rho a} d\mathbf{U} \cdot \mathbf{n}$
- $d\alpha_E = dE - \mathbf{U} \cdot d\mathbf{U} + P d\left(\frac{1}{\rho}\right)$
 a denotes the sound speed

Mean value linearization

- $\alpha_{t,h}^c = \mathbf{U}_h^c \cdot \mathbf{t}$
- $\alpha_{-,h}^c = \left(\frac{1}{\rho}\right)_h^c - \frac{1}{Z_c} \mathbf{U}_h^c \cdot \mathbf{n}$
- $\alpha_{+,h}^c = \left(\frac{1}{\rho}\right)_h^c + \frac{1}{Z_c} \mathbf{U}_h^c \cdot \mathbf{n}$
- $\alpha_{E,h}^c = E_h^c - \mathbf{U}_0^c \cdot \mathbf{U}_h^c + P_0^c \left(\frac{1}{\rho}\right)_h^c$
 where $Z_c = a_0^c \rho_0^c$

System variables polynomial approximation components

- $\left(\frac{1}{\rho}\right)_k^c = \frac{1}{2}(\alpha_{+,k}^c + \alpha_{-,k}^c)$
- $\mathbf{U}_k^c = \frac{1}{2}Z_c(\alpha_{+,k}^c - \alpha_{-,k}^c)\mathbf{n} + \alpha_{t,k}^c \mathbf{t}$
- $E_k^c = \alpha_{E,k}^c + \frac{1}{2}Z_c(\alpha_{+,k}^c - \alpha_{-,k}^c)\mathbf{U}_0^c \cdot \mathbf{n} + \alpha_{t,k}^c \mathbf{U}_0^c \cdot \mathbf{t} - \frac{1}{2}P_0^c(\alpha_{+,k}^c + \alpha_{-,k}^c)$

Unit direction ensuring symmetry preservation

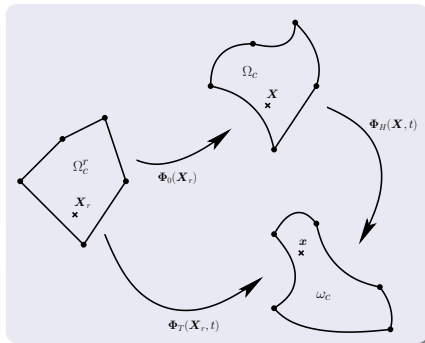
- $\mathbf{n} = \frac{\mathbf{U}_0^c}{\|\mathbf{U}_0^c\|}$ and $\mathbf{t} = \mathbf{e}_z \times \frac{\mathbf{U}_0^c}{\|\mathbf{U}_0^c\|}$

Composed derivatives

- $F_T = \nabla_{X_r} \Phi_T(\mathbf{X}_r, t)$
 $= \nabla_X \Phi_H(\mathbf{X}, t) \circ \nabla_{X_r} \Phi_0(\mathbf{X}_r)$
 $= F_H F_0$
- $J_T(\mathbf{X}_r, t) = J_H(\mathbf{X}, t) J_0(\mathbf{X}_r)$

Mass conservation

- $\rho^0 J_0 = \rho J_T$

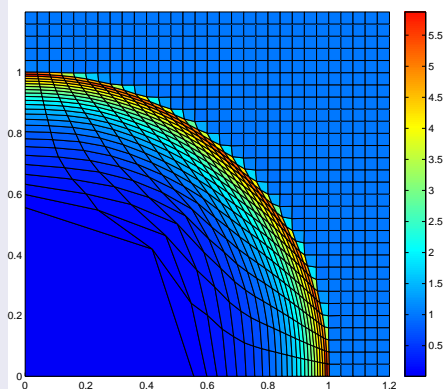


Modification of the mass matrix

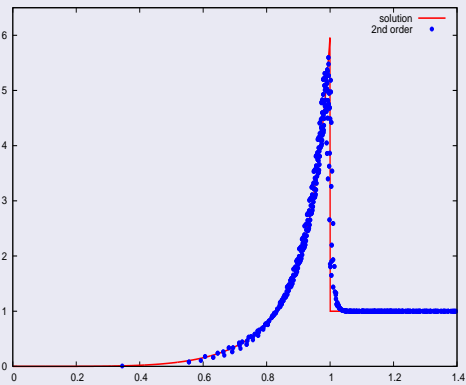
- $\int_{\omega_c} \rho \frac{d\psi_h^c}{dt} \sigma_j d\omega = \sum_{k=0}^K \frac{d\psi_k}{dt} \int_{\Omega_r^c} \rho^0 J_0 \sigma_j \sigma_k d\Omega^r$ time rate of change of successive moments of function ψ
- New definitions of mass matrix, of mass averaged value and of the associated scalar product

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Sedov point blast problem on a Cartesian grid



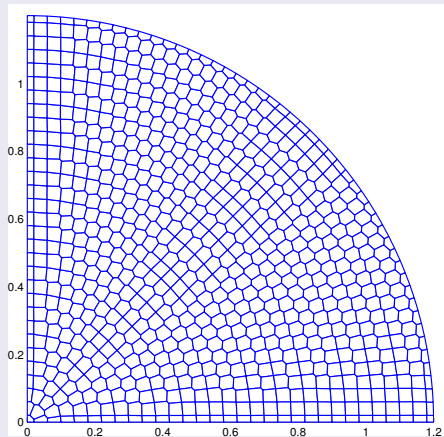
(a) Second-order scheme.



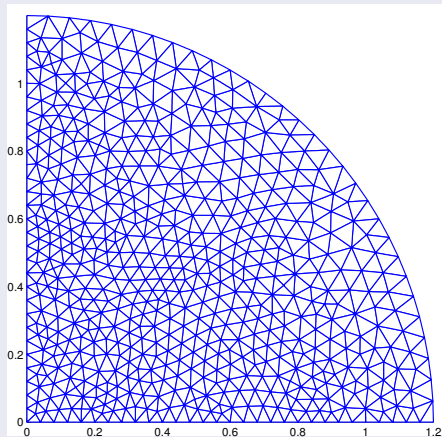
(b) Density profile.

Figure: Point blast Sedov problem on a Cartesian grid made of 30×30 cells: density.

Sedov point blast problem on unstructured grids



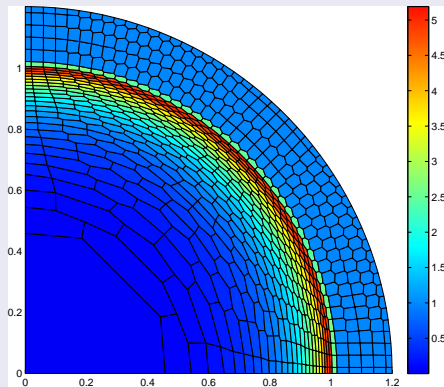
(a) Polygonal grid.



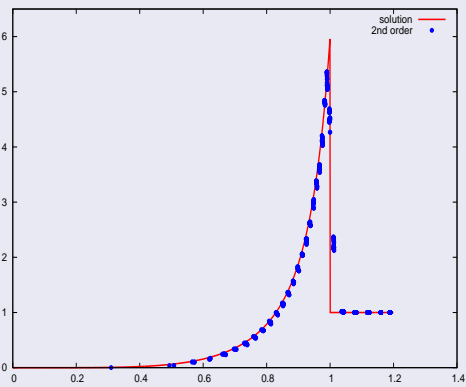
(b) Triangular grid.

Figure: Unstructured initial grids for the point blast Sedov problem.

Sedov point blast problem a polygonal grid



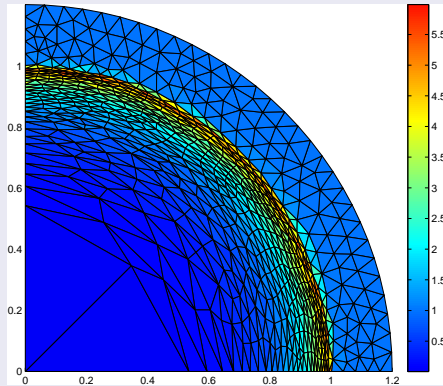
(a) Second-order scheme.



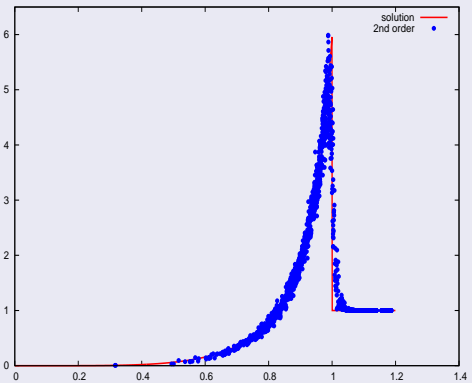
(b) Density profile.

Figure: Point blast Sedov problem on an unstructured grid made of 775 polygonal cells: density map.

Sedov point blast problem on a triangular grid



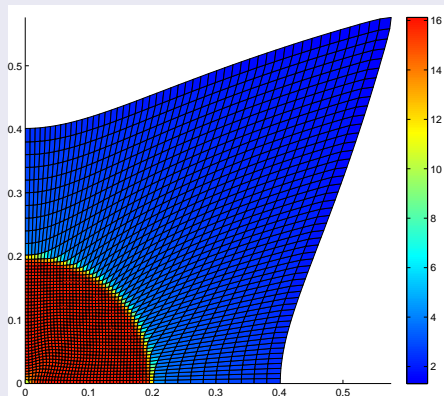
(a) Second-order scheme.



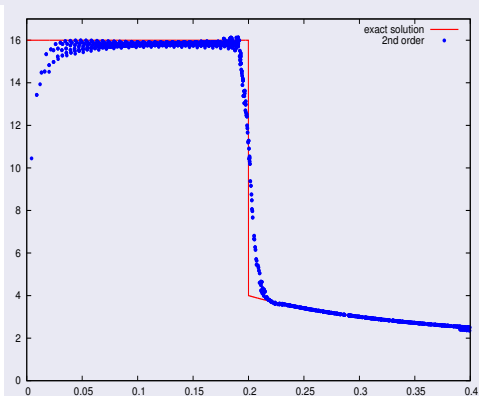
(b) Density profile.

Figure: Point blast Sedov problem on an unstructured grid made of 1100 triangular cells: density map.

Noh problem



(a) Second-order scheme.



(b) Density profile.

Figure: Noh problem on a Cartesian grid made of 50×50 cells: density.

Taylor-Green vortex problem, introduced by R. Rieben (LLNL)

(a) Second-order scheme.

(b) Exact solution.

Figure: Motion of a 10×10 Cartesian mesh through a T.-G. vortex, at $t = 0.75$.

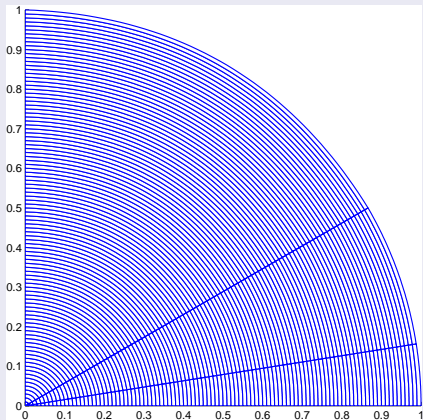
Taylor-Green vortex problem

	L_1		L_2		L_∞	
h	$E_{L_1}^h$	$q_{L_1}^h$	$E_{L_2}^h$	$q_{L_2}^h$	$E_{L_\infty}^h$	$q_{L_\infty}^h$
$\frac{1}{10}$	5.06E-3	1.94	6.16E-3	1.93	2.20E-2	1.84
$\frac{1}{20}$	1.32E-3	1.98	1.62E-3	1.97	5.91E-3	1.95
$\frac{1}{40}$	3.33E-4	1.99	4.12E-4	1.99	1.53E-3	1.98
$\frac{1}{80}$	8.35E-5	2.00	1.04E-4	2.00	3.86E-4	1.99
$\frac{1}{160}$	2.09E-5	-	2.60E-5	-	9.69E-5	-

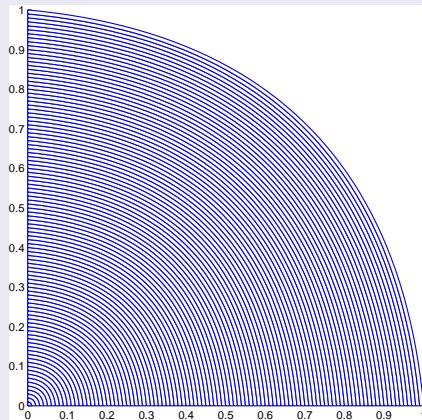
Table: Rate of convergence computed on the pressure at time $t = 0.1$.

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Polar grids



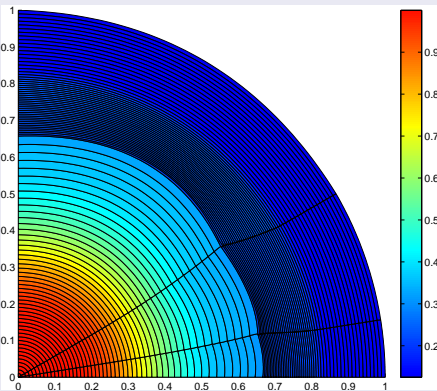
(a) Non-uniform grid.



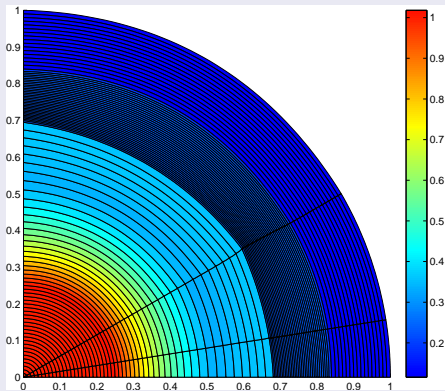
(b) One angular cell grid.

Figure: Polar initial grids for the Sod shock tube problem.

Symmetry preservation



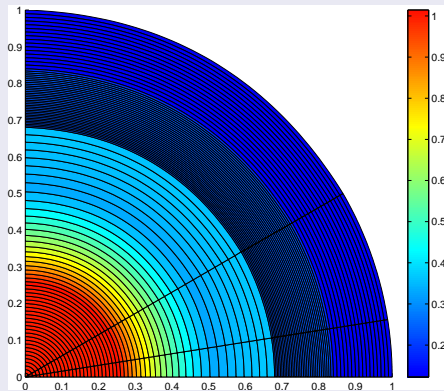
(a) First-order scheme.



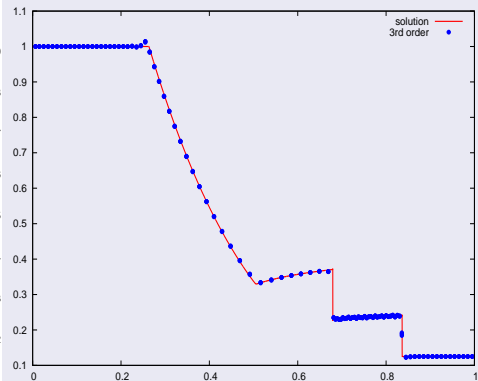
(b) Second-order scheme.

Figure: Sod shock tube problem on a polar grid made of 100×3 non-uniform cells.

Symmetry preservation



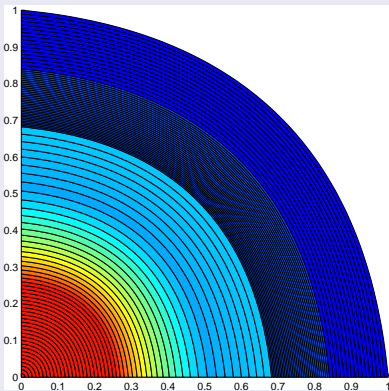
(a) Density map.



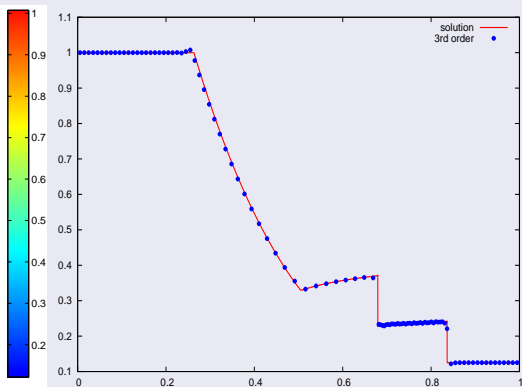
(b) Density profile.

Figure: Third-order DG solution for a Sod shock tube problem on a polar grid made of 100×3 non-uniform cells.

One angular cell polar Sod shock tube problem



(a) Density map.



(b) Density profile.

Figure: Third-order DG solution for a Sod shock tube problem on a polar grid made of 100×1 cells.

Variant of the incompressible Gresho vortex problem

(a) First-order scheme.

(b) Second-order scheme.

Figure: Motion of a polar grid defined in polar coordinates by $(r, \theta) \in [0, 1] \times [0, 2\pi]$, with 40×18 cells at $t = 1$: zoom on the zone $(r, \theta) \in [0, 0.5] \times [0, 2\pi]$.

Variant of the incompressible Gresho vortex problem

(a) Third-order scheme.

(b) Exact solution.

Figure: Motion of a polar grid defined in polar coordinates by $(r, \theta) \in [0, 1] \times [0, 2\pi]$, with 40×18 cells at $t = 1$: zoom on the zone $(r, \theta) \in [0, 0.5] \times [0, 2\pi]$.

Variant of the Gresho vortex problem

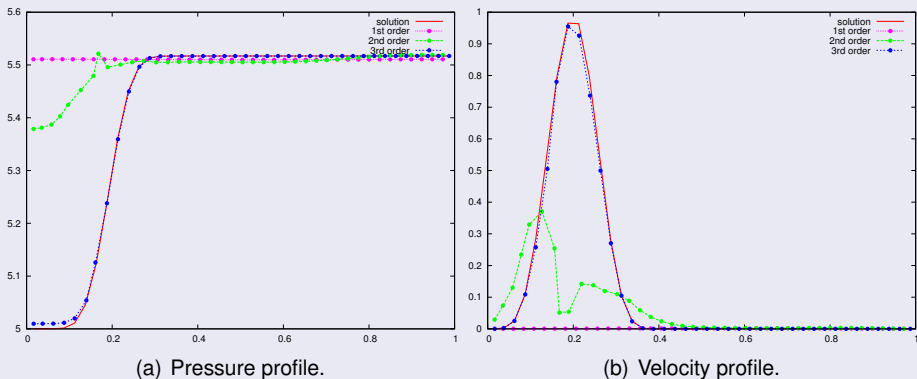


Figure: Gresho variant problem on a polar grid defined in polar coordinates by $(r, \theta) \in [0, 1] \times [0, 2\pi]$, with 40×18 cells at $t = 1$.

Kidder isentropic compression

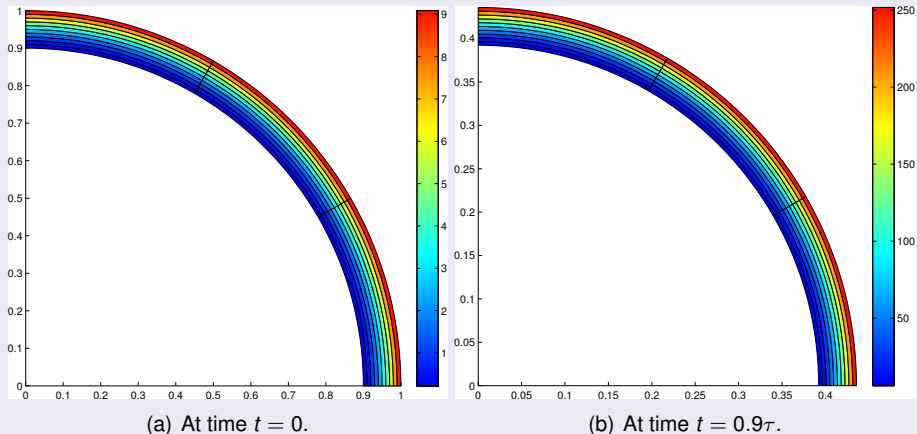


Figure: Third-order DG solution for a Kidder isentropic compression problem on a polar grid made of 10×3 cells: pressure map.

Kidder isentropic compression

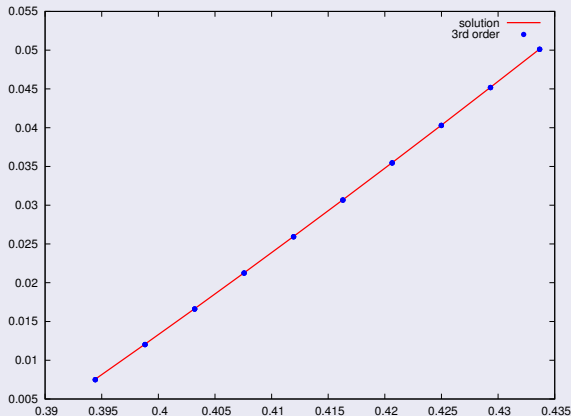


Figure: Third-order DG solution for a Kidder isentropic compression problem on a polar grid made of 10×3 cells: density profile.

Taylor-Green vortex problem

(a) Third-order scheme.

(b) Exact solution.

Figure: Motion of a 10×10 Cartesian mesh through a T.-G. vortex, at $t = 0.75$.

Taylor-Green vortex problem

h	L_1		L_2		L_∞	
	$E_{L_1}^h$	$q_{L_1}^h$	$E_{L_2}^h$	$q_{L_2}^h$	$E_{L_\infty}^h$	$q_{L_\infty}^h$
$\frac{1}{10}$	2.67E-4	2.96	3.36E-4	2.94	1.21E-3	2.86
$\frac{1}{20}$	3.43E-5	2.97	4.36E-5	2.96	1.66E-4	2.93
$\frac{1}{40}$	4.37E-6	2.99	5.59E-6	2.98	2.18E-5	2.96
$\frac{1}{80}$	5.50E-7	2.99	7.06E-7	2.99	2.80E-6	2.99
$\frac{1}{160}$	6.91E-8	-	8.87E-8	-	3.53E-7	-

Table: Rate of convergence computed on the pressure at time $t = 0.1$.

Taylor-Green vortex problem

D.O.F	N	$E_{L_1}^h$	$E_{L_2}^h$	$E_{L_\infty}^h$	time (sec)
600	24×25	2.67E-2	3.31E-2	8.55E-2	2.01
2400	48×50	1.36E-2	1.69E-2	4.37E-2	11.0

Table: First-order DG scheme at time $t = 0.1$.

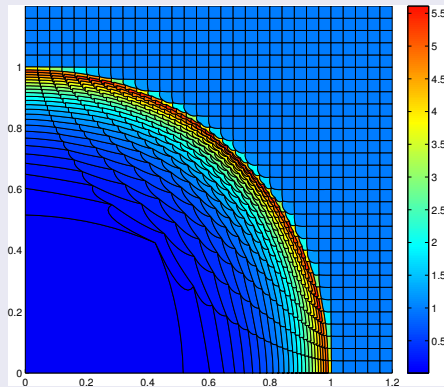
D.O.F	N	$E_{L_1}^h$	$E_{L_2}^h$	$E_{L_\infty}^h$	time (sec)
630	14×15	2.76E-3	3.33E-3	1.07E-2	2.77
2436	28×29	7.52E-4	9.02E-4	2.73E-3	11.3

Table: Second-order DG scheme without limitation at time $t = 0.1$.

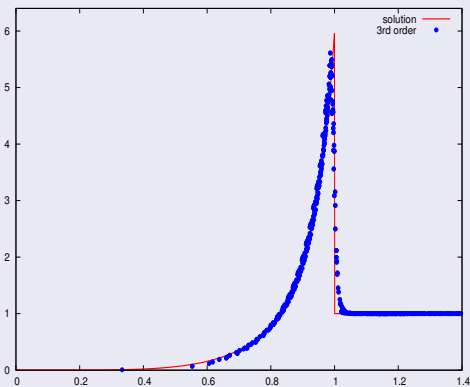
D.O.F	N	$E_{L_1}^h$	$E_{L_2}^h$	$E_{L_\infty}^h$	time (sec)
600	10×10	2.67E-4	3.36E-4	1.21E-3	4.00
2400	20×20	3.43E-5	4.36E-5	1.66E-4	30.6

Table: Third-order DG scheme without limitation at time $t = 0.1$.

Sedov point blast problem on a Cartesian grid



(a) Third-order scheme.



(b) Density profile.

Figure: Point blast Sedov problem on a Cartesian grid made of 30×30 cells: density.

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Conclusions

- Development of 2nd and 3rd order DG schemes for the 2D gas dynamics system in a total Lagrangian formalism
- **GCL and Piola compatibility condition ensured by construction**
- **Dramatic improvement of symmetry preservation by means of third-order DG scheme**
- Riemann invariants limitation

Perspectives

- **High-order limitation**
 - Positivity preserving limitation
 - WENO limiter
- Code parallelization
- Development of a 3rd order DG scheme on moving mesh
- Extension to 3D
- Extension to ALE and solid dynamics



F. VILAR, P.-H. MAIRE AND R. ABGRALL, *Cell-centered discontinuous Galerkin discretizations for two-dimensional scalar conservation laws on unstructured grids and for one-dimensional Lagrangian hydrodynamics*. Computers and Fluids, 2010.



F. VILAR, *Cell-centered discontinuous Galerkin discretization for two-dimensional Lagrangian hydrodynamics*. Computers and Fluids, 2012.



F. VILAR, P.-H. MAIRE AND R. ABGRALL, *Third order Cell-Centered DG scheme for Lagrangian hydrodynamics on general unstructured Bezier grids*. Article in preparation.