Positivity-preserving Lagrangian schemes

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Eulerian formalism (spatial description)

- Fixed referential attached to the observer
- Fixed observation area in which the fluid flows through

Lagrangian formalism (material description)

- Moving referential attached to the material
- Observation area getting moved and deformed through the fluid flow

Advantages of the Lagrangian formalism

- Adapted to the study of regions undergoing large shape changes
- Naturally tracks interfaces in multimaterial compressible flows
- No numerical diffusion from the discretization of the convection terms

Disadvantages of the Lagrangian formalism

- **Robustness issue in cases of shear flows or vortexes**

$\Longrightarrow$ ALE (Arbitrary Lagrangian-Eulerian) method

Definitions

- ρ is the fluid density
- u is the fluid velocity
- e is the fluid specific total energy
- p is the fluid pressure
- $\varepsilon = e - \frac{1}{2}u^2$ is the fluid specific internal energy

Euler equations

- $\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$ Continuity equation
- $\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} = 0$ Momentum conservation equation
- $\frac{\partial \rho e}{\partial t} + \frac{\partial (\rho u e + p u)}{\partial x} = 0$ Total energy conservation equation

Thermodynamical closure

- $p = p(\rho, \varepsilon)$ Equation of state (EOS)

Momentum equation

- $\dfrac{\partial \rho u}{\partial t} + \dfrac{\partial (\rho u^2 + p)}{\partial x} = 0$
- $\rho (\dfrac{\partial u}{\partial t} + u \dfrac{\partial u}{\partial x}) + u \underbrace{(\dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u}{\partial x})}_{=0} + \dfrac{\partial p}{\partial x} = 0$
- $\rho (\dfrac{\partial u}{\partial t} + u \dfrac{\partial u}{\partial x}) + \dfrac{\partial p}{\partial x} = 0$

Total energy equation

- $\dfrac{\partial \rho e}{\partial t} + \dfrac{\partial (\rho u e + p u)}{\partial x} = 0$
- $\rho (\dfrac{\partial e}{\partial t} + u \dfrac{\partial e}{\partial x}) + e \underbrace{(\dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u}{\partial x})}_{=0} + \dfrac{\partial p u}{\partial x} = 0$
- $\rho (\dfrac{\partial e}{\partial t} + u \dfrac{\partial e}{\partial x}) + \dfrac{\partial p u}{\partial x} = 0$

Definitions

- $\tau = \frac{1}{\rho}$ is the specific volume
- $\mathrm{U} = (\tau, u, e)^{\mathrm{t}}$ is the variables vector
- $\mathrm{F}(\mathrm{U}) = (-u, p, p\,u)^{\mathrm{t}}$ is the flux vector

Continuity equation

- $\dfrac{\partial\,\rho}{\partial t} + \dfrac{\partial\,\rho\,u}{\partial x} = 0$
- $\dfrac{\partial\,\rho}{\partial t} + u\,\dfrac{\partial\,\rho}{\partial x} + \rho\,\dfrac{\partial\,u}{\partial x} = 0$
- $\rho\,(\dfrac{\partial\,\tau}{\partial t} + u\,\dfrac{\partial\,\tau}{\partial x}) - \dfrac{\partial\,u}{\partial x} = 0$

Gas dynamics equations

- $\rho\,(\dfrac{\partial\,\mathrm{U}}{\partial t} + u\,\dfrac{\partial\,\mathrm{U}}{\partial x}) + \dfrac{\partial\,\mathrm{F}(\mathrm{U})}{\partial x} = 0$

Moving referential

- X is the position of a point of the fluid in its initial configuration
- $x(X, t)$ is the actual position of this point advected through the fluid flow

Trajectory equation

- $\dfrac{\partial x(X,t)}{\partial t} = u(x(X,t),t)$
- $x(X,0) = X$

Material derivative

- $f(x,t)$ is a fluid variable with sufficient smoothness
- $\dfrac{\mathrm{d} f}{\mathrm{d} t} \equiv \dfrac{\partial f(x(X,t),t)}{\partial t} = \dfrac{\partial f}{\partial t} + u\,\dfrac{\partial f}{\partial x}$

Updated Lagrangian formulation

- $\rho \dfrac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \dfrac{\partial\,\mathrm{F}(\mathrm{U})}{\partial x} = 0$ Moving configuration

Definitions

- $J = \frac{\partial x}{\partial X}$ is the Jacobian associated to the fluid flow
- $\rho^0(X)$ is the initial fluid density

Mass conservation

- $\int_{\omega(0)} \rho^0 \,\mathrm{d}X = \int_{\omega(t)} \rho \,\mathrm{d}x$
- $\int_{\omega(t)} \rho \,\mathrm{d}x = \int_{\omega(0)} \rho\, J \,\mathrm{d}X$
- $\rho\, J = \rho^0$

Total Lagrangian formulation

- $\rho^0 \dfrac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \dfrac{\partial\,\mathrm{F}(\mathrm{U})}{\partial X} = 0$ Fixed configuration

Definitions

- $\mathrm{d}m = \rho\,\mathrm{d}x = \rho^0\,\mathrm{d}X$ is the mass variable
- $\mathrm{A}(\mathrm{U}) = \dfrac{\partial \mathrm{F}(\mathrm{U})}{\partial \mathrm{U}}$ is the Jacobian matrix of the flux
- $a = a(\rho, \varepsilon)$ is the sound speed

Conservative formulation

- $\dfrac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \dfrac{\partial\,\mathrm{F}(\mathrm{U})}{\partial m} = 0$

Non-conservative formulation

- $\dfrac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \mathrm{A}(\mathrm{U})\,\dfrac{\partial\,\mathrm{U}}{\partial m} = 0$
- $\lambda(\mathrm{U}) = \{-\rho\,a\,,\,0\,,\,\rho\,a\}$ are the eigenvalues of matrix A(U)

Definitions

- $0 = t^0 < t^1 < \cdots < t^N = T$ is a partition of the time domain $[0, T]$
- $\Delta t^n = t^{n+1} - t^n$ is the n^{th} time step
- $\omega^0 = \bigcup_{i=1,I} \omega_i^0$ is a partition of the initial computational domain ω^0
- $\omega_i^0 = [X_{i-\frac{1}{2}}, X_{i+\frac{1}{2}}]$ is a generic cell of size ΔX_i
- $\omega_i^n = [x_{i-\frac{1}{2}}^n, x_{i+\frac{1}{2}}^n]$ is the image at time t^n of ω_i^0 through the flow map
- $m_i = \rho_i^0 \, \Delta X_i = \rho_i^n \, \Delta x_i^n$ is the constant mass of cell ω_i
- $\mathrm{U}_i^n = (\tau_i^n, u_i^n, e_i^n)^{\mathrm{t}}$ is the discrete solution vector

First-order finite volume scheme

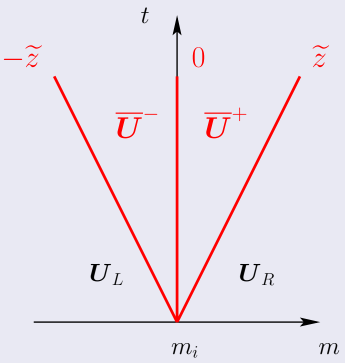
- $\mathrm{U}_i^{n+1} = \mathrm{U}_i^n - \frac{\Delta t^n}{m_i}(\overline{\mathrm{F}}_{i+\frac{1}{2}}^n - \overline{\mathrm{F}}_{i-\frac{1}{2}}^n)$
- $x_{i+\frac{1}{2}}^{n+1} = x_{i+\frac{1}{2}}^n + \Delta t^n \, \overline{u}_{i+\frac{1}{2}}^n$

Numerical fluxes

- $\overline{\mathrm{F}}_{i+\frac{1}{2}}^n = (-\overline{u}_{i+\frac{1}{2}}^n \, , \overline{p}_{i+\frac{1}{2}}^n \, , \overline{p}_{i+\frac{1}{2}}^n \, \overline{u}_{i+\frac{1}{2}}^n)^{\mathrm{t}}$

One state linearization

- $\frac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \mathrm{A}(\mathrm{U})\,\frac{\partial\,\mathrm{U}}{\partial m} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \mathrm{A}(\widetilde{\mathrm{U}})\,\frac{\partial\,\mathrm{U}}{\partial m} = 0$



Approximate Riemann fan

Simple Riemann problem

- $\mathrm{U}(m,0) = \begin{cases} \mathrm{U}_L & \text{for } m\text{-}m_i < 0 \\ \mathrm{U}_R & \text{for } m\text{-}m_i > 0 \end{cases}$
- $\mathrm{U}(m,t) = \begin{cases} \mathrm{U}_L & \text{for } m\text{-}m_i < \text{-}\widetilde{z}\,t \\ \overline{\mathrm{U}}^{-} & \text{for } \text{-}\widetilde{z}\,t < m\text{-}m_i < 0 \\ \overline{\mathrm{U}}^{+} & \text{for } \widetilde{z}\,t > m\text{-}m_i > 0 \\ \mathrm{U}_R & \text{for } m\text{-}m_i > \widetilde{z}\,t \end{cases}$

Relations

- $\widetilde{z} = \widetilde{\rho\, a} > 0$
- $\overline{u}^{-} = \overline{u}^{+} = \overline{u}, \quad \overline{p}^{-} = \overline{p}^{+} = \overline{p}$

Numerical fluxes

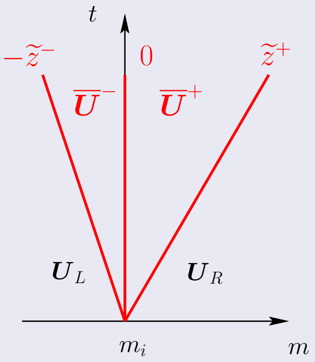
- $\overline{u} = \dfrac{u_L + u_R}{2} - \dfrac{1}{2\,\widetilde{z}}(p_R - p_L)$
- $\overline{p} = \dfrac{p_L + p_R}{2} - \dfrac{\widetilde{z}}{2}(u_R - u_L)$

Intermediate states

- $\overline{\tau}^- = \tau_L + \dfrac{\overline{u} - u_L}{\widetilde{z}}$
- $\overline{\tau}^+ = \tau_R - \dfrac{\overline{u} - u_R}{\widetilde{z}}$
- $\overline{e}^- = e_L - \dfrac{\overline{p}\,\overline{u} - p_L\,u_L}{\widetilde{z}}$
- $\overline{e}^+ = e_R + \dfrac{\overline{p}\,\overline{u} - p_R\,u_R}{\widetilde{z}}$

Two states linearization

- $\dfrac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \mathrm{A}(\mathrm{U})\,\dfrac{\partial\,\mathrm{U}}{\partial m} = 0 \quad \Longrightarrow \quad \begin{cases} \dfrac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \mathrm{A}(\widetilde{\mathrm{U}_L})\,\dfrac{\partial\,\mathrm{U}}{\partial m} = 0 & \text{for } m\text{-}m_i < 0 \\ \dfrac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \mathrm{A}(\widetilde{\mathrm{U}_R})\,\dfrac{\partial\,\mathrm{U}}{\partial m} = 0 & \text{for } m\text{-}m_i > 0 \end{cases}$



Approximate Riemann fan

Simple Riemann problem

- $\mathrm{U}(m,0) = \begin{cases} \mathrm{U}_L & \text{for } m\text{-}m_i < 0 \\ \mathrm{U}_R & \text{for } m\text{-}m_i > 0 \end{cases}$
- $\mathrm{U}(m,t) = \begin{cases} \mathrm{U}_L & \text{for } m\text{-}m_i < \text{-}\widetilde{z}^-\, t \\ \overline{\mathrm{U}}^- & \text{for } \text{-}\widetilde{z}^-\, t < m\text{-}m_i < 0 \\ \overline{\mathrm{U}}^+ & \text{for } \widetilde{z}^+\, t > m\text{-}m_i > 0 \\ \mathrm{U}_R & \text{for } m\text{-}m_i > \widetilde{z}^+\, t \end{cases}$

Relations

- $\widetilde{z}^- = \widetilde{\rho\, a}^- > 0, \qquad \widetilde{z}^+ = \widetilde{\rho\, a}^+ > 0$
- $\overline{u}^- = \overline{u}^+ = \overline{u}, \qquad \overline{p}^- = \overline{p}^+ = \overline{p}$

Numerical fluxes

- $\overline{u} = \dfrac{\widetilde{z}^- \, u_L + \widetilde{z}^+ \, u_R}{\widetilde{z}^- + \widetilde{z}^+} - \dfrac{1}{\widetilde{z}^- + \widetilde{z}^+}(p_R - p_L)$
- $\overline{p} = \dfrac{\widetilde{z}^+ \, p_L + \widetilde{z}^- \, p_R}{\widetilde{z}^- + \widetilde{z}^+} - \dfrac{\widetilde{z}^- \, \widetilde{z}^+}{\widetilde{z}^- + \widetilde{z}^+}(u_R - u_L)$

Intermediate states

- $\overline{\tau}^- = \tau_L + \dfrac{\overline{u} - u_L}{\widetilde{z}^-}$ and $\overline{\tau}^+ = \tau_R - \dfrac{\overline{u} - u_R}{\widetilde{z}^+}$
- $\overline{e}^- = e_L - \dfrac{\overline{p}\,\overline{u} - p_L\,u_L}{\widetilde{z}^-}$ and $\overline{e}^+ = e_R + \dfrac{\overline{p}\,\overline{u} - p_R\,u_R}{\widetilde{z}^+}$

Acoustic solver

- $\widetilde{z}^- \equiv z_L^n = \rho_L^n \, a_L^n$ Left acoustic impedance
- $\widetilde{z}^+ \equiv z_R^n = \rho_R^n \, a_R^n$ Right acoustic impedance

Convex combination

- $\mathrm{U}_i^{n+1} = \mathrm{U}_i^n - \frac{\Delta t^n}{m_i}(\overline{\mathrm{F}}_{i+\frac{1}{2}}^n - \overline{\mathrm{F}}_{i-\frac{1}{2}}^n) \pm \frac{\Delta t^n}{m_i}\mathrm{F}(\mathrm{U}_i^n) \pm \frac{\Delta t^n}{m_i}(\widetilde{z}_{i-\frac{1}{2}}^+ + \widetilde{z}_{i+\frac{1}{2}}^-)\mathrm{U}_i^n$
- $\mathrm{U}_i^{n+1} = (1-\lambda_i)\mathrm{U}_i^n + \lambda_{i+\frac{1}{2}}^- \overline{\mathrm{U}}_{i+\frac{1}{2}}^- + \lambda_{i-\frac{1}{2}}^+ \overline{\mathrm{U}}_{i-\frac{1}{2}}^+$

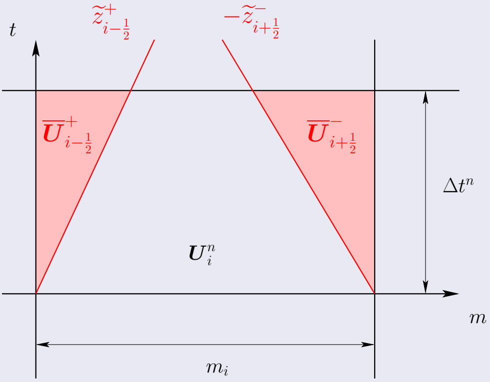


illustration of the Lagrangian scheme

Definitions

- $\lambda_{i\pm\frac{1}{2}}^{\mp} = \frac{\Delta t^n}{m_i}\widetilde{z}_{i\pm\frac{1}{2}}^{\mp}$
- $\lambda_i = \lambda_{i+\frac{1}{2}}^- + \lambda_{i-\frac{1}{2}}^+$
- $\overline{\mathrm{U}}_{i\pm\frac{1}{2}}^{\mp} = \mathrm{U}_i^n \mp \dfrac{\overline{\mathrm{F}}_{i\pm\frac{1}{2}}^n - \mathrm{F}(\mathrm{U}_i^n)}{\widetilde{z}_{i\pm\frac{1}{2}}^{\mp}}$

CFL condition: $\lambda_i \leq 1$

- $\Delta t^n \leq \dfrac{m_i}{\widetilde{z}_{i+\frac{1}{2}}^- + \widetilde{z}_{i-\frac{1}{2}}^+}$
- $\Delta t^n \leq \frac{1}{2}\dfrac{\Delta x_i^n}{a_i^n}$ if $\widetilde{z}_{i\pm\frac{1}{2}}^{\mp} = z_i^n$

Ideal EOS for the perfect gas

- $P = \rho\,(\gamma - 1)\,\varepsilon$ where $a = \sqrt{\frac{\gamma\,P}{\rho}}$
- If $\rho > 0$ then $\varepsilon > 0 \iff a^2 > 0 \quad (\Leftrightarrow\ P > 0)$

Stiffened EOS for water

- $P = \rho\,(\gamma - 1)\,\varepsilon - \gamma\,P^\star$ where $a = \sqrt{\frac{\gamma\,(P+P^\star)}{\rho}}$
- If $\rho > 0$ then $\rho\varepsilon > P^\star \iff a^2 > 0 \quad (\Leftrightarrow\ P > -P^\star)$

Jones-Wilkins-Lee (JWL) EOS for the detonation-products gas

- $P = \rho\,(\gamma - 1)\,\varepsilon + f(\rho)$ where $a = \sqrt{\frac{\gamma\,P - f(\rho) + \rho\,f'(\rho)}{\rho}}$
- If $\rho > 0$ then $\varepsilon > 0 \implies a^2 > 0 \quad (\Leftrightarrow\ P > f(\rho) \geq 0)$

Mie-Grüneisen EOS for solids

- $P = \rho_0\,\Gamma_0\,\varepsilon + \rho_0\,a_0^2\,f(\rho)$ where $a = \sqrt{a_0^2\,f'(\eta) + \frac{\rho_0\,\Gamma_0\,p}{\rho^2}}$
- If $\rho \in [\rho^\star, \frac{S_m}{S_m - 1}\,\rho_0[$ then $\varepsilon > 0 \implies a^2 > 0 \quad (\Leftrightarrow\ p > \rho_0\,a_0^2\,f(\rho))$

Requirements

- $\Delta x_i^n > 0 \iff \tau_i^n > 0$ — Positive volume and density
- $(a_i^n)^2 = (a(\mathrm{U}_i^n))^2 > 0$ — Computable sound speed

Convex admissible set

- $$G = \left\{ \mathrm{U} = \begin{pmatrix} \tau \\ u \\ e \end{pmatrix}, \quad \tau \in]\tau_{min}, \tau_{max}[\text{ and } \varepsilon(\mathrm{U}) = e - \frac{1}{2}u^2 > \varepsilon_{min} \right\}$$

Positivity-preserving scheme

- Under which constraint, $\mathrm{U}_i^n \in G$ does imply $\mathrm{U}_i^{n+1} \in G$
- Numerical fluxes definition?
- CFL condition?
- something else?

Convex combination

- $\mathrm{U}_i^{n+1} = (1-\lambda_i)\mathrm{U}_i^n + \lambda_{i+\frac{1}{2}}^{-}\,\overline{\mathrm{U}}_{i+\frac{1}{2}}^{-} + \lambda_{i-\frac{1}{2}}^{+}\,\overline{\mathrm{U}}_{i-\frac{1}{2}}^{+}$
- If $\mathrm{U}_i^n \in G$ then $\overline{\mathrm{U}}_{i\pm\frac{1}{2}}^{\mp} \in G \implies \mathrm{U}_i^{n+1} \in G$

Intermediate specific volume

- $\overline{\tau}_{i\pm\frac{1}{2}}^{\mp} = \tau_i^n \pm \dfrac{\overline{u}_{i\pm\frac{1}{2}} - u_i^n}{\widetilde{z}_{i\pm\frac{1}{2}}^{\mp}} = \tau_i^n \pm \dfrac{\tau_i^n\, z_i^n}{\widetilde{z}_{i\pm\frac{1}{2}}^{\mp}} \underbrace{\left(\dfrac{\overline{u}_{i\pm\frac{1}{2}} - u_i^n}{a_i^n}\right)}_{v_{i\pm\frac{1}{2}}}$
- $\overline{\tau}_{i\pm\frac{1}{2}}^{\mp} = \tau_i^n \left(1 \pm \dfrac{z_i^n}{\widetilde{z}_{i\pm\frac{1}{2}}^{\mp}}\, v_{i\pm\frac{1}{2}}\right)$

Intermediate internal energy

- $\overline{\varepsilon}_{i\pm\frac{1}{2}}^{\mp} = \overline{e}_{i\pm\frac{1}{2}}^{\mp} - \frac{1}{2}(\overline{u}_{i\pm\frac{1}{2}})^2$
- $\overline{\varepsilon}_{i\pm\frac{1}{2}}^{\mp} = \varepsilon_i^n \left(1 \mp \dfrac{\tau_i^n\, p_i^n}{\varepsilon_i^n}\, \dfrac{z_i^n}{\widetilde{z}_{i\pm\frac{1}{2}}^{\mp}}\, v_{i\pm\frac{1}{2}} + \dfrac{1}{2}\, \dfrac{(a_i^n)^2}{\varepsilon_i^n}\, v_{i\pm\frac{1}{2}}^2\right)$

The acoustic solver is not positivity-preserving !

- If $v_{i+\frac{1}{2}} > 1 \iff \overline{u}_{i+\frac{1}{2}} > u_i^n + a_i^n$ then $\overline{\tau}^-_{i+\frac{1}{2}} < 0$

Non-linear Dukowicz solver

- $\widetilde{z}^{\mp}_{i\pm\frac{1}{2}} = z_i^n \left(1 + \Gamma_i \,|v_{i\pm\frac{1}{2}}|\right)$
- $\Gamma_i \geq 1$ is determined in the limit of infinite strength shock wave
- $\lim\limits_{\overline{u}\to\infty} \dfrac{\overline{\tau}^+}{\tau_R} = \dfrac{\gamma-1}{\gamma+1} \implies \Gamma_i = \dfrac{\gamma+1}{2}$ in the ideal gas case

J. K. DUKOWICZ, *A general, non-iterative Riemann solver for Godunov's method.* J. Comput. Phys., 61:119-137, 1984.

Positive intermediate states in the ideal gas case

- $\overline{\tau}^{\mp}_{i\pm\frac{1}{2}} = \tau_i^n \left(1 \pm \dfrac{v_{i\pm\frac{1}{2}}}{1 + \Gamma_i \,|v_{i\pm\frac{1}{2}}|}\right) > 0$
- $\overline{\varepsilon}^{\mp}_{i\pm\frac{1}{2}} = \varepsilon_i^n \left(1 \mp (\gamma-1) \dfrac{v_{i\pm\frac{1}{2}}}{1 + \Gamma_i \,|v_{i\pm\frac{1}{2}}|} + \dfrac{1}{2}\gamma(\gamma-1) v^2_{i\pm\frac{1}{2}}\right) > 0$

General case: $\tau \in]\tau_{min}, \tau_{max}[$ and $\varepsilon > \varepsilon_{min}$

- $\widetilde{z}^{\mp}_{i\pm\frac{1}{2}} = z_i^n \,(1 + \widetilde{\Gamma}\,|v_{i\pm\frac{1}{2}}|)$
- $\widetilde{\Gamma} = \sigma_v^{-1} > 0$ is determined to ensure the scheme positive

$\tau > \tau_{min} \geq 0$

- $\overline{\tau}^{\mp}_{i\pm\frac{1}{2}} - \tau_{min} = (\tau_i^n - \tau_{min}) \left(1 \pm \left(\dfrac{\tau_i^n}{\tau_i^n - \tau_{min}} \right) \dfrac{v_{i\pm\frac{1}{2}}}{1 + \widetilde{\Gamma}\,|v_{i\pm\frac{1}{2}}|} \right)$
- $\overline{\tau}^{\mp}_{i\pm\frac{1}{2}} - \tau_{min} > (\tau_i^n - \tau_{min}) \left(1 - \left(\dfrac{\tau_i^n}{\tau_i^n - \tau_{min}} \right) \sigma_v \right)$
- If $\tau_i^n > \tau_{min}$ and $\sigma_v \leq 1 - \frac{\tau_{min}}{\tau_i^n}$ then $\overline{\tau}^{\mp}_{i\pm\frac{1}{2}} > \tau_{min}$

$\tau < \tau_{max} \leq +\infty$

- If $\tau_i^n < \tau_{max}$ and $\sigma_v \leq \frac{\tau_{max}}{\tau_i^n} - 1$ then $\overline{\tau}^{\mp}_{i\pm\frac{1}{2}} < \tau_{max}$

$\varepsilon > \varepsilon_{min} \geq 0$

- $\overline{\varepsilon}^{\mp}_{i\pm\frac{1}{2}} - \varepsilon_{min} = (\varepsilon_i^n - \varepsilon_{min})\, A_i + B_i$
- $A_i = 1 \mp \left(\dfrac{\varepsilon_i^n}{\varepsilon_i^n - \varepsilon_{min}} \right) \dfrac{\tau_i^n\, p_i^n}{\varepsilon_i^n} \dfrac{v_{i\pm\frac{1}{2}}}{1 + \widetilde{\Gamma}\, |v_{i\pm\frac{1}{2}}|}$
- $B_i = \frac{1}{2}(a_i^n)^2\, v^2_{i\pm\frac{1}{2}} \geq 0$
- If $\quad \varepsilon_i^n > \varepsilon_{min} \quad$ and $\quad \sigma_v \leq (1 - \frac{\varepsilon_{min}}{\varepsilon_i^n}) \left| \frac{\rho_i^n\, \varepsilon_i^n}{p_i^n} \right| \quad$ then $\quad \overline{\varepsilon}^{\mp}_{i\pm\frac{1}{2}} > \varepsilon_{min}$

Positivity-preserving finite volume scheme

Finally, the scheme will be positivity-preserving under the following conditions

- $\Delta t^n \leq \dfrac{m_i}{\widetilde{z}^{-}_{i+\frac{1}{2}} + \widetilde{z}^{+}_{i-\frac{1}{2}}}$
- $\widetilde{z}^{\mp}_{i\pm\frac{1}{2}} = z_i^n \left(1 + \sigma_v^{-1} \left| \dfrac{\overline{u}_{i\pm\frac{1}{2}} - u_i^n}{a_i^n} \right| \right)$
- $\sigma_v \leq \min \left(1 - \frac{\tau_{min}}{\tau_i^n}, \frac{\tau_{max}}{\tau_i^n} - 1, (1 - \frac{\varepsilon_{min}}{\varepsilon_i^n}) \left| \frac{\rho_i^n\, \varepsilon_i^n}{p_i^n} \right| \right)$

Implicit definition of the approximate Riemann solution

- $\widetilde{z}^- = \widetilde{z}^-(\mathrm{U}_L, \overline{u}) = z_L\,(1 + \sigma_v^{-1}\,|\frac{\overline{u}-u_L}{a_L}|)$
- $\widetilde{z}^+ = \widetilde{z}^+(\mathrm{U}_R, \overline{u}) = z_R\,(1 + \sigma_v^{-1}\,|\frac{\overline{u}-u_R}{a_R}|)$
- $\overline{u} = \overline{u}(\mathrm{U}_L, \mathrm{U}_R, \widetilde{z}^+, \widetilde{z}^-) = \frac{\widetilde{z}^-\,u_L + \widetilde{z}^+\,u_R}{\widetilde{z}^- + \widetilde{z}^+} - \frac{p_R - p_L}{\widetilde{z}^- + \widetilde{z}^+}$

$\Longrightarrow$ use of an iterative method (fixed point method)

Acoustic solver or generic waves speed definition ?

- $\widetilde{z}^{\mp}_{i\pm\frac{1}{2}} > 0 \quad (= z_i^n$ if we want)
- $\Delta t^n \leq \sigma_e\,\dfrac{m_i}{\widetilde{z}^+_{i-\frac{1}{2}} + \widetilde{z}^-_{i+\frac{1}{2}}}$
- $\Delta t^n < \sigma_v\,\dfrac{\Delta x_i^n}{|\overline{u}_{i+\frac{1}{2}} - \overline{u}_{i-\frac{1}{2}}|} \quad \Longleftrightarrow \quad \dfrac{|\Delta x_i^{n+1} - \Delta x_i^n|}{\Delta x_i^n} < \sigma_v$

$\tau_i^{n+1} > \tau_{min}$

- $\tau_i^{n+1} = \tau_i^n + \frac{\Delta t^n}{m_i}(\overline{u}^n_{i+\frac{1}{2}} - \overline{u}^n_{i-\frac{1}{2}})$
- $\tau_i^{n+1} - \tau_{min} = (\tau_i^n - \tau_{min}) \left(1 + \frac{\Delta t^n}{\Delta x_i^n} \left(\frac{\tau_i^n}{\tau_i^n - \tau_{min}}\right) (\overline{u}^n_{i+\frac{1}{2}} - \overline{u}^n_{i-\frac{1}{2}})\right)$
- $\frac{\Delta t^n}{\Delta x_i^n} |\overline{u}^n_{i+\frac{1}{2}} - \overline{u}^n_{i-\frac{1}{2}}| < \sigma_v$
- $\tau_i^{n+1} - \tau_{min} > (\tau_i^n - \tau_{min}) \left(1 - \left(\frac{\tau_i^n}{\tau_i^n - \tau_{min}}\right) \sigma_v\right)$
- If $\tau_i^n > \tau_{min}$ and $\sigma_v \leq 1 - \frac{\tau_{min}}{\tau_i^n}$ then $\tau_i^{n+1} > \tau_{min}$

$\tau_i^{n+1} < \tau_{max}$

- If $\tau_i^n < \tau_{max}$ and $\sigma_v \leq \frac{\tau_{max}}{\tau_i^n} - 1$ then $\tau_i^{n+1} < \tau_{max}$

$\varepsilon_i^{n+1} > \varepsilon_{min}$

- $\overline{\varepsilon}_i^{n+1} - \varepsilon_{min} = (\varepsilon_i^n - \varepsilon_{min})\, A_i + B_i$
- $A_i = 1 - \frac{\Delta t^n}{\Delta x_i^n} \left(\dfrac{\varepsilon_i^n}{\varepsilon_i^n - \varepsilon_{min}} \right) \dfrac{\tau_i^n\, p_i^n}{\varepsilon_i^n} \,(\overline{u}^n_{i+\frac{1}{2}} - \overline{u}^n_{i-\frac{1}{2}})$
- $B_i = \frac{\Delta t}{m_i} \left[\widetilde{z}^-_{i+\frac{1}{2}}\, w^2_{i+\frac{1}{2}} + \widetilde{z}^+_{i-\frac{1}{2}}\, w^2_{i-\frac{1}{2}} - \frac{\Delta t}{2m_i} (\widetilde{z}^-_{i+\frac{1}{2}}\, w_{i+\frac{1}{2}} + \widetilde{z}^+_{i-\frac{1}{2}}\, w_{i-\frac{1}{2}})^2 \right]$
- $w_{i\pm\frac{1}{2}} = \overline{u}^n_{i\pm\frac{1}{2}} - u_i^n$

$A_i > 0$

- If $\quad \varepsilon_i^n > \varepsilon_{min} \quad$ and $\quad \sigma_v \leq (1 - \frac{\varepsilon_{min}}{\varepsilon_i^n}) \left| \frac{\rho_i^n\, \varepsilon_i^n}{p_i^n} \right| \quad$ then $\quad A_i > 0$

$B_i \geq 0$

- $B_i = \dfrac{\Delta t}{m_i} \begin{pmatrix} \widetilde{z}^+_{i-\frac{1}{2}}(1 - \frac{\Delta t^n}{2m_i}\, \widetilde{z}^+_{i-\frac{1}{2}}) & -\frac{\Delta t^n}{2m_i}\, \widetilde{z}^+_{i-\frac{1}{2}}\, \widetilde{z}^-_{i+\frac{1}{2}} \\ -\frac{\Delta t^n}{2m_i}\, \widetilde{z}^+_{i-\frac{1}{2}}\, \widetilde{z}^-_{i+\frac{1}{2}} & \widetilde{z}^-_{i+\frac{1}{2}}(1 - \frac{\Delta t^n}{2m_i}\, \widetilde{z}^-_{i+\frac{1}{2}}) \end{pmatrix} \begin{pmatrix} w_{i-\frac{1}{2}} \\ w_{i+\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} w_{i-\frac{1}{2}} \\ w_{i+\frac{1}{2}} \end{pmatrix}$
- $B_i \equiv \dfrac{\Delta t}{m_i}\, \mathrm{M}_i\, \mathrm{W} \,.\, \mathrm{W}$
- M_i is positive semi-definite if and only if

$$\Delta t^n \leq 2\, \frac{m_i}{\widetilde{z}^-_{i+\frac{1}{2}} + \widetilde{z}^+_{i-\frac{1}{2}}} \qquad \left(= \frac{\Delta x^n_i}{a^n_i} \qquad \text{if } \widetilde{z}^{\mp}_{i\pm\frac{1}{2}} = z^n_i \right)$$

$\varepsilon_i^{n+1} > \varepsilon_{min}$

- If $\varepsilon_i^n > \varepsilon_{min}$, $\sigma_v \leq (1 - \frac{\tau_{min}}{\tau_i^n}) \left| \frac{\rho_i^n\, \varepsilon_i^n}{p_i^n} \right|$ and $\sigma_e \leq 2$ then $\varepsilon_i^{n+1} > \varepsilon_{min}$

Positivity-preserving finite volume scheme

Finally, the scheme will be positivity-preserving under the following conditions

- $\Delta t^n \leq \sigma_e \, \dfrac{m_i}{\widetilde{z}^-_{i+\frac{1}{2}} + \widetilde{z}^+_{i-\frac{1}{2}}}$
- $\sigma_e \leq 2$
- $\Delta t^n < \sigma_v \, \dfrac{\Delta x_i^n}{|\overline{u}^n_{i+\frac{1}{2}} - \overline{u}^n_{i-\frac{1}{2}}|}$
- $\sigma_v \leq \min\left(1 - \frac{\tau_{min}}{\tau_i^n}, \frac{\tau_{max}}{\tau_i^n} - 1, (1 - \frac{\varepsilon_{min}}{\varepsilon_i^n})\,|\frac{\rho_i^n\,\varepsilon_i^n}{p_i^n}|\right)$

Less or more constraining ?

- $\widetilde{z}^-_{i+\frac{1}{2}} + \widetilde{z}^+_{i-\frac{1}{2}} = \frac{m_i}{\Delta x_i^n}\left[2a_i^n + \sigma_v^{-1}(|\overline{u}^n_{i+\frac{1}{2}} - u_i^n| + |\overline{u}^n_{i-\frac{1}{2}} - u_i^n|)\right]$
- $\widetilde{z}^-_{i+\frac{1}{2}} + \widetilde{z}^+_{i-\frac{1}{2}} > \frac{m_i}{\Delta x_i^n\,\sigma_v}|\overline{u}^n_{i+\frac{1}{2}} - \overline{u}^n_{i-\frac{1}{2}}|$

$$\Longrightarrow \quad \frac{m_i}{\widetilde{z}^-_{i+\frac{1}{2}} + \widetilde{z}^+_{i-\frac{1}{2}}} < \sigma_v \, \frac{\Delta x_i^n}{|\overline{u}^n_{i+\frac{1}{2}} - \overline{u}^n_{i-\frac{1}{2}}|}$$

Euler equations

- $\dfrac{\partial \rho}{\partial t} + \nabla_x \centerdot (\rho\, \boldsymbol{u}) = 0$ Continuity equation
- $\dfrac{\partial \rho\, \boldsymbol{u}}{\partial t} + \nabla_x \centerdot (\rho\, \boldsymbol{u} \otimes \boldsymbol{u} + p\mathrm{I}_d) = \boldsymbol{0}$ Momentum conservation equation
- $\dfrac{\partial \rho\, e}{\partial t} + \nabla_x \centerdot (\rho\, \boldsymbol{u}\, e + p\, \boldsymbol{u}) = 0$ Total energy conservation equation

Trajectory equation

- $\dfrac{\mathrm{d}\, \boldsymbol{x}(\boldsymbol{X}, t)}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{X}, t), t), \quad \boldsymbol{x}(\boldsymbol{X}, 0) = \boldsymbol{X}$

Material time derivative

- $\dfrac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{x}, t) = \dfrac{\partial}{\partial t} f(\boldsymbol{x}, t) + \boldsymbol{u} \centerdot \nabla_x f(\boldsymbol{x}, t)$

Definitions

- $\mathsf{U} = (\tau, \boldsymbol{u}, e)^{\mathrm{t}}$
- $\mathsf{F}(\mathsf{U}) = (-\boldsymbol{u},\ \mathbb{1}(1)\, p,\ \mathbb{1}(2)\, p,\ p\, \boldsymbol{u})^{\mathrm{t}}$
- $\mathbb{1}(i)_j = (\delta_{i1}, \delta_{i2})^{\mathrm{t}}$

Updated Lagrangian formulation

- $\rho \dfrac{\mathrm{d}\,\mathsf{U}}{\mathrm{d}t} + \nabla_x \boldsymbol{.}\, \mathsf{F}(\mathsf{U}) = 0$ Moving configuration

Deformation gradient tensor

- $\mathsf{J} = \nabla_X \boldsymbol{x}$ and $|\mathsf{J}| = \det \mathsf{J} > 0$

Mass conservation

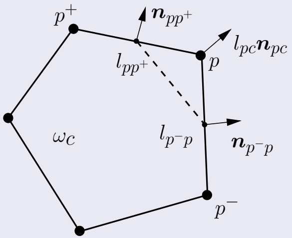
- $\rho\, |\mathsf{J}| = \rho^0$

Total Lagrangian formulation

- $\rho^0 \dfrac{\mathrm{d}\,\mathsf{U}}{\mathrm{d}t} + \nabla_X \boldsymbol{.}\, (|\mathsf{J}|\mathsf{J}^{-1}\mathsf{F}(\mathsf{U})) = 0$ Fixed configuration

Definitions

- $\mathcal{P}_c = \mathcal{P}(\omega_c)$ Node set
- $l_{pc}^- \boldsymbol{n}_{pc}^- = \frac{1}{2} l_{p^-p} \boldsymbol{n}_{p^-p}$ Right corner normal
- $l_{pc}^+ \boldsymbol{n}_{pc}^+ = \frac{1}{2} l_{pp^+} \boldsymbol{n}_{pp^+}$ Left corner normal
- $l_{pc} \boldsymbol{n}_{pc} = l_{pc}^- \boldsymbol{n}_{pc}^- + l_{pc}^+ \boldsymbol{n}_{pc}^+$ Corner normal
- $\mathrm{U}_c = \dfrac{1}{m_c} \displaystyle\int_{\omega_c} \rho\, \mathrm{U}\, \mathrm{d}v$ Discrete solution vector



Cell notations

Discretization

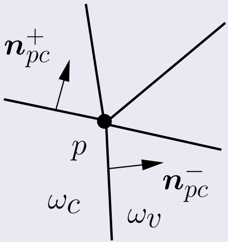
- $\mathrm{U}_c^{n+1} = \mathrm{U}_c^n - \dfrac{\Delta t^n}{m_c} \displaystyle\int_{\partial\omega_c} \overline{\mathrm{F}} \,.\, \boldsymbol{n}\, \mathrm{d}l = \mathrm{U}_c^n - \dfrac{\Delta t^n}{m_c} \sum_{p\in\mathcal{P}_c} \dfrac{1}{2} \left(\overline{\mathrm{F}}_{pc}^+ + \overline{\mathrm{F}}_{p^+c}^- \right) \,.\, l_{pp^+} \boldsymbol{n}_{pp^+}$
- $\overline{\mathrm{F}}_{pc}^\pm = (-\overline{\boldsymbol{u}}_p,\ \mathbb{1}(1)\, \overline{p}_{pc}^\pm,\ \mathbb{1}(2)\, \overline{p}_{pc}^\pm,\ \overline{p}_{pc}^\pm\, \overline{\boldsymbol{u}}_p)^{\mathrm{t}}$ Numerical fluxes

First-order finite volume scheme

- $\mathrm{U}_c^{n+1} = \mathrm{U}_c^n - \dfrac{\Delta t^n}{m_c} \displaystyle\sum_{p\in\mathcal{P}_c} \left(\overline{\mathrm{F}}_{pc}^- \,.\, l_{pc}^- \boldsymbol{n}_{pc}^- + \overline{\mathrm{F}}_{pc}^+ \,.\, l_{pc}^+ \boldsymbol{n}_{pc}^+ \right)$

1D Riemann solver

- $\mathcal{C}_p = \mathcal{C}(p)$ Neighboring cells set
- $\overline{\boldsymbol{u}}_p \,.\, \boldsymbol{n}_{pc}^- = \boldsymbol{u}_i^n \,.\, \boldsymbol{n}_{pc}^- - \dfrac{\overline{p}_{pc}^- - p_c^n}{\widetilde{z}_{pc}^-}$
- $\overline{\boldsymbol{u}}_p \,.\, \boldsymbol{n}_{pc}^+ = \boldsymbol{u}_i^n \,.\, \boldsymbol{n}_{pc}^+ - \dfrac{\overline{p}_{pc}^+ - p_c^n}{\widetilde{z}_{pc}^+}$



Point notations

Conservation

- $\sum_c m_c \mathrm{U}_c^{n+1} = \sum_c m_c \mathrm{U}_c^n$ If no boundary condition
- $\sum_{c \in \mathcal{C}_p} \left(\overline{p}_{pc}^- \, l_{pc}^- \boldsymbol{n}_{pc}^- + \overline{p}_{pc}^+ \, l_{pc}^+ \boldsymbol{n}_{pc}^+ \right) = \mathbf{0}$
- $\sum_{c \in \mathcal{C}_p} p_c^n \, l_{pc} \boldsymbol{n}_{pc} - \Big(\underbrace{\widetilde{z}_{pc}^- l_{pc}^- (\boldsymbol{n}_{pc}^- \otimes \boldsymbol{n}_{pc}^-) + \widetilde{z}_{pc}^+ l_{pc}^+ (\boldsymbol{n}_{pc}^+ \otimes \boldsymbol{n}_{pc}^+)}_{\mathrm{M}_{pc}} \Big) (\overline{\boldsymbol{u}}_p - \boldsymbol{u}_c^n) = \mathbf{0}$

Nodal solver

- $\overline{\boldsymbol{u}}_p = \Big(\sum\limits_{c \in \mathcal{C}_p} \mathrm{M}_{pc} \Big)^{-1} \sum\limits_{c \in \mathcal{C}_p} \Big(\mathrm{M}_{pc} \boldsymbol{u}_c^n + P_c^n \, l_{pc} \boldsymbol{n}_{pc} \Big)$
- $\overline{p}_{pc}^{\pm} = p_c^n - \widetilde{z}_{pc}^{\pm} (\overline{\boldsymbol{u}}_p - \boldsymbol{u}_c^n) \cdot \boldsymbol{n}_{pc}^{\pm}$

Convex combination

- $\mathrm{U}_c^{n+1} = \mathrm{U}_c^n - \dfrac{\Delta t^n}{m_c} \sum\limits_{p \in \mathcal{P}_c} \Big(\overline{\mathrm{F}}_{pc}^{-} \cdot l_{pc}^{-} \boldsymbol{n}_{pc}^{-} + \overline{\mathrm{F}}_{pc}^{+} \cdot l_{pc}^{+} \boldsymbol{n}_{pc}^{+} \Big) + \dfrac{\Delta t^n}{m_c} \mathrm{F}(\mathrm{U}_c^n) \cdot \sum\limits_{p \in \mathcal{P}_c} l_{pc}^{\pm} \boldsymbol{n}_{pc}^{\pm}$
- $\mathrm{U}_c^{n+1} = (1 - \lambda_c) \mathrm{U}_c^n + \sum\limits_{p \in \mathcal{P}_c} \lambda_{pc}^{-} \overline{\mathrm{U}}_{pc}^{-} + \sum\limits_{p \in \mathcal{P}_c} \lambda_{pc}^{+} \overline{\mathrm{U}}_{pc}^{+}$
- $\lambda_{pc}^{\pm} = \dfrac{\Delta t^n}{m_c} \widetilde{z}_{pc}^{\pm} l_{pc}^{\pm}$
- $\lambda_c = \sum\limits_{p \in \mathcal{P}_c} (\lambda_{pc}^{-} + \lambda_{pc}^{+})$
- $\overline{\mathrm{U}}_{pc}^{\pm} = \mathrm{U}_c^n - \dfrac{(\overline{\mathrm{F}}_{pc}^{\pm} - \mathrm{F}(\mathrm{U}_c^n)) \cdot \boldsymbol{n}_{pc}^{\pm}}{\widetilde{z}_{pc}^{\pm}}$

Convex admissible set

- $G = \left\{ \mathrm{U} = (\tau, \boldsymbol{u}, e)^{\mathrm{t}}, \quad \tau \in]\tau_{min}, \tau_{max}[\text{ and } \varepsilon(\mathrm{U}) = e - \frac{1}{2}\boldsymbol{u}^2 > \varepsilon_{min} \right\}$

Convex combination

- $\mathrm{U}_c^{n+1} = (1 - \lambda_c)\mathrm{U}_c^n + \sum_{p \in \mathcal{P}_c} \lambda_{pc}^{-} \overline{\mathrm{U}}_{pc}^{-} + \sum_{p} \lambda_{pc}^{+} \overline{\mathrm{U}}_{pc}^{+}$
- $\Delta t^n \leq \dfrac{m_c}{\sum_p (\widetilde{z}_{pc}^{-} l_{pc}^{-} + \widetilde{z}_{pc}^{+} l_{pc}^{+})} \qquad \left(= \dfrac{|\omega_c^n|}{a_c^n \sum_p l_{pp^+}} \qquad \text{if} \quad \widetilde{z}_{pc}^{\pm} = z_c^n \right)$

Intermediate states

- $\overline{\tau}_{pc}^{\pm} = \tau_c^n + \dfrac{(\overline{\boldsymbol{u}}_p - \boldsymbol{u}_c^n) \,.\, \boldsymbol{n}_{pc}^{\pm}}{\widetilde{z}_{pc}^{\pm}}$
- $\overline{\boldsymbol{u}}_{pc}^{\pm} = \boldsymbol{u}_c^n - \dfrac{(\overline{p}_{pc}^{\pm} - p_c^n)\, \boldsymbol{n}_{pc}^{\pm}}{\widetilde{z}_{pc}^{\pm}} \neq \overline{\boldsymbol{u}}_p$
- $\overline{e}_{pc}^{\pm} = e_c^n - \dfrac{(\overline{p}_{pc}^{\pm}\, \overline{\boldsymbol{u}}_p - p_c^n\, \boldsymbol{u}_c^n) \,.\, \boldsymbol{n}_{pc}^{\pm}}{\widetilde{z}_{pc}^{\pm}}$

Intermediate specific volume

- $\overline{\tau}_{pc}^{\pm} = \tau_c^n + \dfrac{(\overline{\boldsymbol{u}}_p - \boldsymbol{u}_c^n) \cdot \boldsymbol{n}_{pc}^{\pm}}{\widetilde{z}_{pc}^{\pm}} = \tau_c^n + \dfrac{\tau_c^n \, z_c^n}{\widetilde{z}_{pc}^{\pm}} \underbrace{\dfrac{(\overline{\boldsymbol{u}}_p - \boldsymbol{u}_c^n) \cdot \boldsymbol{n}_{pc}^{\pm}}{a_c^n}}_{v_{pc}^{\pm}}$
- $\overline{\tau}_{pc}^{\pm} = \tau_c^n \left(1 + \dfrac{z_c^n}{\widetilde{z}_{pc}^{\pm}} v_{pc}^{\pm}\right)$ similar to the 1D $\overline{\tau}_{i\pm\frac{1}{2}}^{\mp} = \tau_i^n \left(1 \pm \dfrac{z_i^n}{\widetilde{z}_{i\pm\frac{1}{2}}^{\mp}} v_{i\pm\frac{1}{2}}\right)$

Modified non-linear Dukowicz solver

- $\widetilde{z}_{pc}^{\pm} = z_c^n \, \left(1 + \sigma_v^{-1} \, |v_{pc}^{\pm}|\right)$

$\tau \in \,]\tau_{min}, \tau_{max}[$

- If $\tau_c^n > \tau_{min}$ and $\sigma_v \leq 1 - \frac{\tau_{min}}{\tau_c^n}$ then $\overline{\tau}_{pc}^{\pm} > \tau_{min}$
- If $\tau_c^n < \tau_{max}$ and $\sigma_v \leq \frac{\tau_{max}}{\tau_c^n} - 1$ then $\overline{\tau}_{pc}^{\pm} < \tau_{max}$

Intermediate velocity

- $\overline{\boldsymbol{u}}_{pc}^{\pm} \neq \overline{\boldsymbol{u}}_{p}$
- $\overline{\boldsymbol{u}}_{pc}^{\pm} \,.\, \boldsymbol{n}_{pc}^{\pm} = \overline{\boldsymbol{u}}_{p} \,.\, \boldsymbol{n}_{pc}^{\pm}$
- $\overline{\boldsymbol{u}}_{pc}^{\pm} \,.\, \boldsymbol{t}_{pc}^{\pm} = \boldsymbol{u}_{c}^{n} \,.\, \boldsymbol{t}_{pc}^{\pm}$

Intermediate internal energy

- $\overline{\varepsilon}_{pc}^{\pm} = \overline{e}_{pc}^{\pm} - \frac{1}{2}(\overline{\boldsymbol{u}}_{pc}^{\pm})^2$
- $\overline{\varepsilon}_{pc}^{\pm} = \varepsilon_c^n - p_c^n \tau_c^n \dfrac{z_c^n}{\widetilde{z}_{pc}^{\pm}}\, v_{pc}^{\pm} + \dfrac{1}{2}(a_c^n\, v_{pc}^{\pm})^2$
- $\overline{\varepsilon}_{i\pm\frac{1}{2}}^{\mp} = \varepsilon_i^n \mp p_i^n \tau_i^n \dfrac{z_i^n}{\widetilde{z}_{i\pm\frac{1}{2}}^{\mp}}\, v_{i\pm\frac{1}{2}} + \dfrac{1}{2}(a_i^n\, v_{i\pm\frac{1}{2}})^2$ in the 1D case

$\varepsilon > \varepsilon_{min}$

- If $\varepsilon_c^n > \varepsilon_{min}$ and $\sigma_v \leq (1 - \frac{\varepsilon_{min}}{\varepsilon_c^n}) \left| \frac{\rho_c^n \varepsilon_c^n}{p_c^n} \right|$ then $\overline{\varepsilon}_{pc}^{\pm} > \varepsilon_{min}$

Positivity-preserving finite volume scheme

Finally, the scheme will be positivity-preserving under the following conditions

- $\Delta t^n \leq \dfrac{m_c}{\sum_p (\widetilde{z}_{pc}^- l_{pc}^- + \widetilde{z}_{pc}^+ l_{pc}^+)}$
- $\widetilde{z}_{pc}^{\pm} = z_c^n \left(1 + \sigma_v^{-1} \left| \dfrac{(\overline{\boldsymbol{u}}_p - \boldsymbol{u}_c^n) \cdot \boldsymbol{n}_{pc}^{\pm}}{a_c^n} \right| \right)$
- $\sigma_v \leq \min \left(1 - \frac{\tau_{min}}{\tau_c^n}, \frac{\tau_{max}}{\tau_c^n} - 1, (1 - \frac{\varepsilon_{min}}{\varepsilon_c^n}) \left| \frac{\rho_c^n \varepsilon_c^n}{p_c^n} \right| \right)$

Acoustic solver or generic waves speed definition ?

- $\widetilde{z}_{pc}^{\pm} > 0 \quad (= z_c^n$ if we want)
- $\Delta t^n \leq \sigma_e \, \dfrac{m_c}{\sum_p (\widetilde{z}_{pc}^- l_{pc}^- + \widetilde{z}_{pc}^+ l_{pc}^+)}$
- $\Delta t^n < \sigma_v \, \dfrac{|\omega_c^n|}{|\sum_p \overline{\boldsymbol{u}}_p \cdot l_{pc} \boldsymbol{n}_{pc}|} \quad \Longleftrightarrow \quad \dfrac{\left| |\omega_c^{n+1}| - |\omega_c^n| \right|}{|\omega_c^n|} < \sigma_v$

$\tau_c^{n+1} > \tau_{min}$

- $$\tau_c^{n+1} = \tau_c^n + \frac{\Delta t^n}{m_c} \sum_{p \in \mathcal{P}_c} \overline{\boldsymbol{u}}_p \,.\, l_{pc} \boldsymbol{n}_{pc}$$
- $$\tau_c^{n+1} - \tau_{min} = (\tau_c^n - \tau_{min}) \left(1 + \frac{\Delta t^n}{|\omega_c^n|} \left(\frac{\tau_c^n}{\tau_c^n - \tau_{min}} \right) \sum_{p \in \mathcal{P}_c} \overline{\boldsymbol{u}}_p \,.\, l_{pc} \boldsymbol{n}_{pc} \right)$$
- $$\frac{\Delta t^n}{|\omega_c^n|} \left| \sum_{p \in \mathcal{P}_c} \overline{\boldsymbol{u}}_p \,.\, l_{pc} \boldsymbol{n}_{pc} \right| < \sigma_v$$
- $$\tau_c^{n+1} - \tau_{min} > (\tau_c^n - \tau_{min}) \left(1 - \left(\frac{\tau_c^n}{\tau_c^n - \tau_{min}} \right) \sigma_v \right)$$
- If $\tau_c^n > \tau_{min}$ and $\sigma_v \leq 1 - \frac{\tau_{min}}{\tau_c^n}$ then $\tau_c^{n+1} > \tau_{min}$

$\tau_c^{n+1} < \tau_{max}$

- If $\tau_c^n < \tau_{max}$ and $\sigma_v \leq \frac{\tau_{max}}{\tau_c^n} - 1$ then $\tau_c^{n+1} < \tau_{max}$

$\varepsilon_c^{n+1} > \varepsilon_{min}$

- $\overline{\varepsilon}_c^{n+1} - \varepsilon_{min} = (\varepsilon_c^n - \varepsilon_{min})\, A_c + B_c$
- $A_c = 1 - \frac{\Delta t^n}{|\omega_c^n|} \left(\dfrac{\varepsilon_c^n}{\varepsilon_c^n - \varepsilon_{min}} \right) \dfrac{\tau_c^n\, p_c^n}{\varepsilon_c^n} \sum_{p \in \mathcal{P}_c} \overline{\boldsymbol{u}}_p \,.\, l_{pc} \boldsymbol{n}_{pc}$
- $B_c = \dfrac{\Delta t^n}{m_c} \left[\sum_{p \in \mathcal{P}_c} \mathsf{M}_{pc} \boldsymbol{w}_p \,.\, \boldsymbol{w}_p - \dfrac{\Delta t^n}{2\, m_c} \left(\sum_{p \in \mathcal{P}_c} \mathsf{M}_{pc} \boldsymbol{w}_p \right)^2 \right]$
- $\boldsymbol{w}_p = \overline{\boldsymbol{u}}_p - u_c^n$

$A_c > 0$

- If $\varepsilon_c^n > \varepsilon_{min}$ and $\sigma_v \leq (1 - \frac{\varepsilon_{min}}{\varepsilon_c^n}) \left| \frac{\rho_c^n\, \varepsilon_c^n}{p_c^n} \right|$ then $A_c > 0$

$B_c \geq 0$

- $$B_c = \frac{\Delta t^n}{m_c}\Big[\sum_{p\in\mathcal{P}_c} \mathrm{M}_{pc}\boldsymbol{w}_p \cdot \boldsymbol{w}_p - \frac{\Delta t^n}{2\,m_c}\Big(\sum_{p\in\mathcal{P}_c} \mathrm{M}_{pc}\boldsymbol{w}_p\Big)^2\Big]$$
- $$\mathrm{M}_{pc} = \widetilde{z}_{pc}^- l_{pc}^-(\boldsymbol{n}_{pc}^- \otimes \boldsymbol{n}_{pc}^-) + \widetilde{z}_{pc}^+ l_{pc}^+(\boldsymbol{n}_{pc}^+ \otimes \boldsymbol{n}_{pc}^+)$$
- $$\sum_{p\in\mathcal{P}_c} \mathrm{M}_{pc}\boldsymbol{w}_p \cdot \boldsymbol{w}_p = \sum_{p\in\mathcal{P}_c} \widetilde{z}_{pc}^-\, l_{pc}^-(\underbrace{\boldsymbol{w}_p \cdot \boldsymbol{n}_{pc}^-}_{Y^-})^2 + \widetilde{z}_{pc}^+\, l_{pc}^+(\underbrace{\boldsymbol{w}_p \cdot \boldsymbol{n}_{pc}^+}_{Y^+})^2$$
- Re-numbering: $\displaystyle\sum_{p\in\mathcal{P}_c} \psi_p^- + \psi_p^+ = \sum_{q=1}^{2|\mathcal{P}_c|} \psi_q$
- $$B_c = \frac{\Delta t^n}{m_c}\Big[\sum_{q=1}^{2|\mathcal{P}_c|} z_q\, l_q\, Y_q^2 - \frac{\Delta t^n}{2\,m_c}\sum_{q,r=1}^{2|\mathcal{P}_c|} z_q z_r\, l_q l_r\, Y_q Y_r\,(\boldsymbol{n}_q \cdot \boldsymbol{n}_r)\Big] \equiv \frac{\Delta t^n}{m_c}\mathrm{H}\boldsymbol{Y} \cdot \boldsymbol{Y}$$
- $\boldsymbol{Y} = (Y_1, \dots, Y_{2|\mathcal{P}_c|})^t$ and $\mathrm{H}_{qr} = \begin{cases} z_q\, l_q\,(1 - \frac{\Delta t^n}{2m_c} z_q\, l_q) & \text{if } q = r \\ -\frac{\Delta t^n}{2m_c} z_q z_r\, l_q l_r\,(\boldsymbol{n}_q \cdot \boldsymbol{n}_r) & \text{if } q \neq r \end{cases}$

Theorem

- If H is symmetric diagonally dominant with non-negative diagonal entries then H is positive semi-definite (thanks to Gerschgorin theorem)

$B_c \geq 0$

- If $\displaystyle \frac{\Delta t^n}{m_c} \leq \frac{2}{z_q\, l_q} = \frac{2}{\widetilde{z}_{pc}^{\pm}\, l_{pc}^{\pm}}$ then $\mathrm{H}_{qq} \geq 0$
- If $\displaystyle \frac{\Delta t^n}{m_c} \leq \frac{2}{\sum_q z_q\, l_q |\boldsymbol{n}_q \,.\, \boldsymbol{n}_r|}$ then $\displaystyle |\mathrm{H}_{qq}| - \sum_{r \neq q} |\mathrm{H}_{qr}| \geq 0$
- If $\displaystyle \frac{\Delta t^n}{m_c} \leq \frac{2}{\sum_q z_q\, l_q} \iff \Delta t^n \leq 2\, \frac{m_c}{\sum_p (\widetilde{z}_{pc}^{-} l_{pc}^{-} + \widetilde{z}_{pc}^{+} l_{pc}^{+})}$ then $B_c \geq 0$

$\varepsilon_c^{n+1} > \varepsilon_{min}$

- If $\varepsilon_c^n > \varepsilon_{min}$, $\sigma_v \leq (1 - \frac{\varepsilon_{min}}{\varepsilon_c^n}) \left| \frac{\rho_c^n\, \varepsilon_c^n}{p_c^n} \right|$ and $\sigma_e \leq 2$ then $\varepsilon_c^{n+1} > \varepsilon_{min}$

Positivity-preserving finite volume scheme

Finally, the scheme will be positivity-preserving under the following conditions

- $\Delta t^n \leq \sigma_e \; \dfrac{m_c}{\sum_p (\widetilde{z}_{pc}^- l_{pc}^- + \widetilde{z}_{pc}^+ l_{pc}^+)}$
- $\sigma_e \leq 2$
- $\Delta t^n < \sigma_v \; \dfrac{|\omega_c^n|}{|\sum_p \overline{\boldsymbol{u}}_p \cdot l_{pc} \boldsymbol{n}_{pc}|}$
- $\sigma_v \leq \min \left(1 - \frac{\tau_{min}}{\tau_c^n}, \frac{\tau_{max}}{\tau_c^n} - 1, (1 - \frac{\varepsilon_{min}}{\varepsilon_c^n}) \, |\frac{\rho_c^n \, \varepsilon_c^n}{p_c^n}| \right)$

Less or more constraining ?

- $\widetilde{z}_{pc}^- l_{pc}^- + \widetilde{z}_{pc}^+ l_{pc}^+ = \frac{m_c}{|\omega_c^n|} \left[2 a_c^n + \sigma_v^{-1} (l_{pc}^- |(\overline{u}_p - u_c^n) \cdot \boldsymbol{n}_{pc}^-| + l_{pc}^+ |(\overline{u}_p - u_c^n) \cdot \boldsymbol{n}_{pc}^+|) \right]$
- $\sum_p (\widetilde{z}_{pc}^- l_{pc}^- + \widetilde{z}_{pc}^+ l_{pc}^+) > \frac{m_c}{|\omega_c^n| \, \sigma_v} |\sum_p \overline{\boldsymbol{u}}_p \cdot l_{pc} \boldsymbol{n}_{pc}|$

$$\Longrightarrow \quad \frac{m_c}{\sum_p (\widetilde{z}_{pc}^- l_{pc}^- + \widetilde{z}_{pc}^+ l_{pc}^+)} < \sigma_v \; \frac{|\omega_c^n|}{|\sum_p \overline{\boldsymbol{u}}_p \cdot l_{pc} \boldsymbol{n}_{pc}|}$$

High-order extension of the numerical scheme

- High-order DG schemes on curvilinear polygonal grids
- **Already presented at Brown University on January 2013**

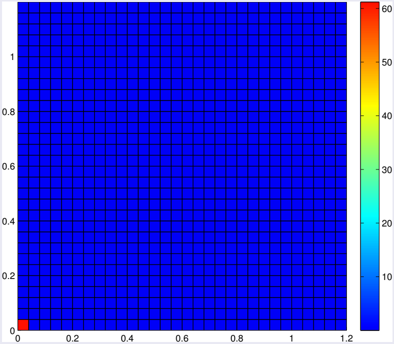
High-order extension of the positivity-preserving proof

- High-order positivity-preserving Lagrangian schemes
- **Already presented at Brown University on March 2014**

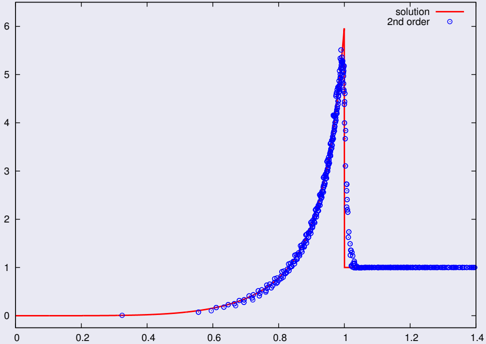
Future work

- Comparison between the different numerical fluxes (acoustic, Dukowicz, modified Dukowicz) in term of accuracy and time step
- End the writing of the article related to this work, started a long time ago already !

Sedov point blast problem on a 30x30 Cartesian grid



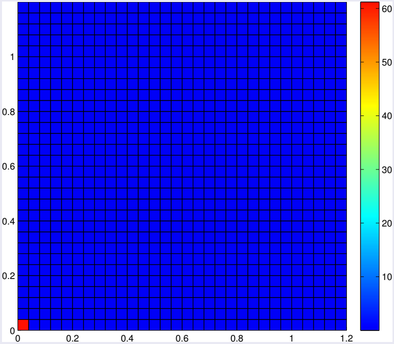
(a) Pressure map



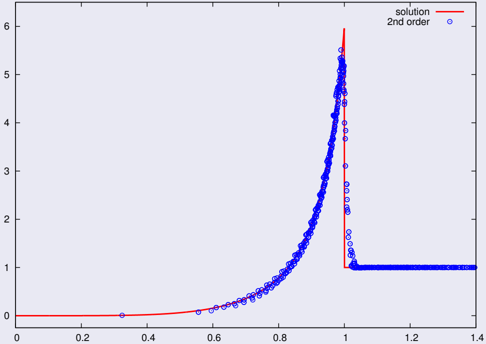
(b) Density profiles

Figure : Second-order DG solution at the final time $t = 1$

Sedov point blast problem on a 30x30 Cartesian grid



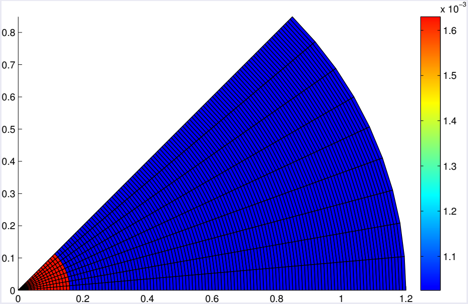
(a) Pressure map



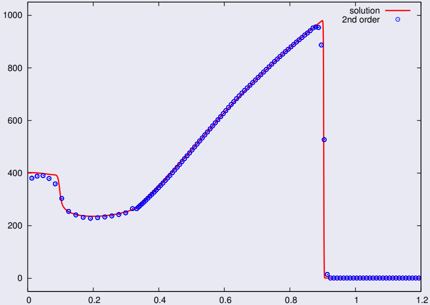
(b) Density profiles

Figure : Second-order DG solution at the final time $t = 1$

Underwater TNT charge explosion on a 120x9 polar grid



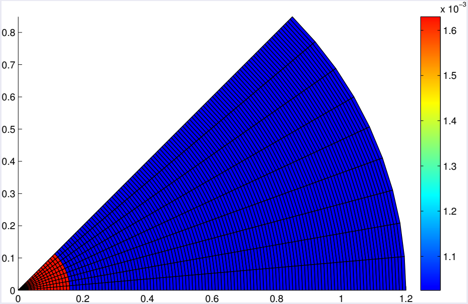
(a) Density map



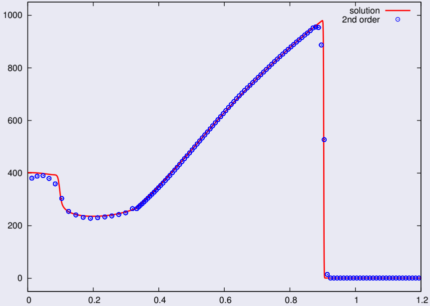
(b) Pressure profiles

Figure : Second-order DG solution at the final time $t = 2.5\text{E-}4$

Underwater TNT charge explosion on a 120x9 polar grid



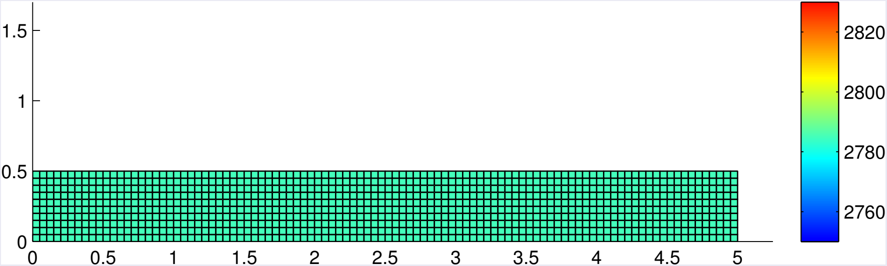
(a) Density map



(b) Pressure profiles

Figure : Second-order DG solution at the final time $t = 2.5\text{E-}4$

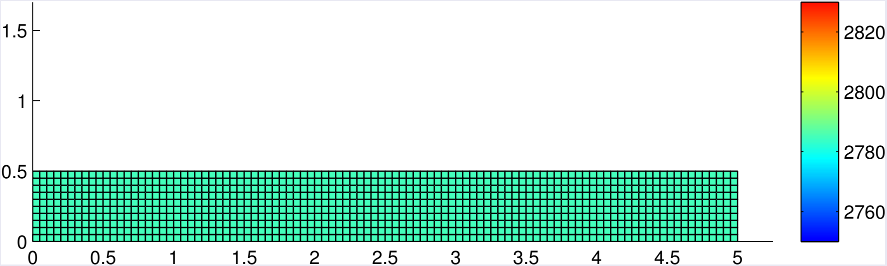
Two-dimensional projectile impact problem on a 100x10 grid



(a) Density map

Figure : Second-order DG solution at the final time $t = 5$E-2

Two-dimensional projectile impact problem on a 100x10 grid



(a) Density map

Figure : Second-order DG solution at the final time $t = 5$E-2