

# *A posteriori* correction of DG schemes through subcell finite volume formulation and flux reconstruction

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- 1 Introduction
- 2 DG as a subcell finite volume
- 3 *A posteriori* subcell correction
- 4 Numerical results
- 5 Conclusion

## History

- Introduced by Reed and Hill in 1973 in the frame of the neutron transport
- Major development and improvements by B. Cockburn and C.-W. Shu in a series of seminal papers

## Procedure

- Local variational formulation
- Piecewise polynomial approximation of the solution in the cells
- Choice of the numerical fluxes
- Time integration

## Advantages

- Natural extension of Finite Volume method
- Excellent analytical properties ( $L_2$  stability,  $hp$ -adaptivity, ...)
- Extremely high accuracy (superconvergent for scalar conservation laws)
- Compact stencil (involve only face neighboring cells)

## 1D scalar conservation law

- $\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0, \quad (x, t) \in \omega \times [0, T]$
- $u(x, 0) = u_0(x), \quad x \in \omega$

## $(k + 1)^{\text{th}}$ order discretization

- $\{\omega_i\}_i$  a partition of  $\omega$ , such that  $\omega_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$
- $0 = t^0 < t^1 < \dots < t^N = T$  a partition of the temporal domain  $[0, T]$
- $u_h(x, t)$  the numerical solution, such that  $u_h|_{\omega_i} = u_h^i \in \mathbb{P}^k(\omega_i)$

$$u_h^i(x, t) = \sum_{m=1}^{k+1} u_m^i(t) \sigma_m(x)$$

- $\{\sigma_m\}_m$  a basis of  $\mathbb{P}^k(\omega_i)$

## Local variational formulation on $\omega_i$

- $\int_{\omega_i} \left( \frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} \right) \psi \, dx = 0 \quad \text{with } \psi(x) \text{ a test function}$

## Integration by parts

$$\bullet \int_{\omega_i} \frac{\partial u}{\partial t} \psi \, dx - \int_{\omega_i} F(u) \frac{\partial \psi}{\partial x} \, dx + \left[ F(u) \psi \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0$$

## Approximated solution

- Substitute  $u$  by  $u_h^i$ , and restrict  $\psi$  to the polynomial space  $\mathbb{P}^k(\omega_i)$

$$\bullet \int_{\omega_i} \frac{\partial u_h^i}{\partial t} \psi \, dx = \int_{\omega_i} F(u_h^i) \frac{\partial \psi}{\partial x} \, dx - \left[ \mathcal{F} \psi \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}, \quad \forall \psi \in \mathbb{P}^k(\omega_i)$$

$$\bullet \sum_{m=1}^{k+1} \frac{\partial u_m^i}{\partial t} \int_{\omega_i} \sigma_m \sigma_p \, dx = \int_{\omega_i} F(u_h^i) \frac{\partial \sigma_p}{\partial x} \, dx - \left[ \mathcal{F} \sigma_p \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}, \quad \forall p \in \llbracket 1, k+1 \rrbracket$$

## Numerical flux

- $\mathcal{F}_{i+\frac{1}{2}} = \mathcal{F} \left( u_h^i(x_{i+\frac{1}{2}}, t), u_h^{i+1}(x_{i+\frac{1}{2}}, t) \right)$
- $\mathcal{F}(u, v) = \frac{F(u) + F(v)}{2} - \frac{\gamma(u, v)}{2} (v - u)$
- $\gamma(u, v) = \max(|F'(u)|, |F'(v)|)$

Local Lax-Friedrichs

# Subcell resolution of DG scheme

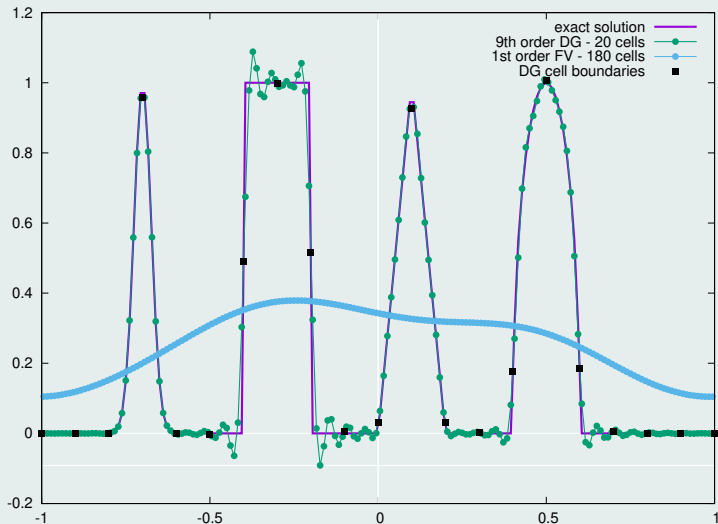


Figure : Linear advection of composite signal after 4 periods

# Subcell resolution of DG scheme

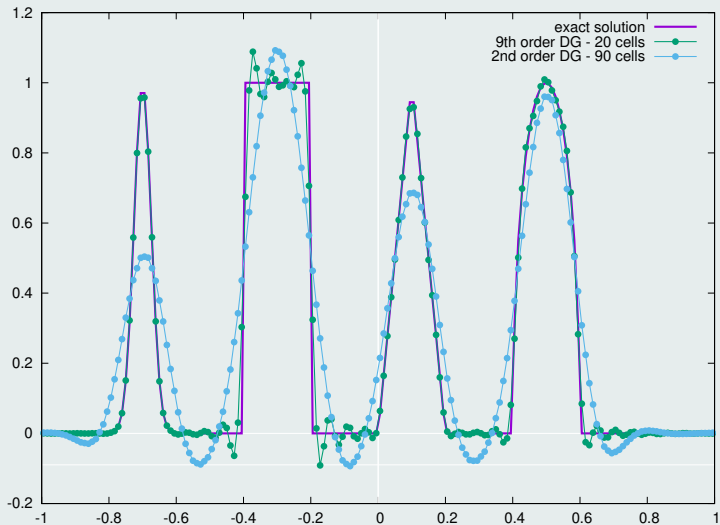


Figure : Linear advection of composite signal after 4 periods

## Gibbs phenomenon

- High-order schemes leads to spurious oscillations near discontinuities
- Leads potentially to nonlinear instability, non-admissible solution, crash
- Vast literature of how prevent this phenomenon to happen:

⇒ *a priori* and ***a posteriori*** limitations

## *A priori* limitation

- Artificial viscosity
- Flux limitation
- Slope/moment limiter
- Hierarchical limiter
- ENO/WENO limiter

## *A posteriori* limitation

- MOOD (“Multi-dimensional Optimal Order Detection”)
- Subcell finite volume limitation
- **Subcell limitation through flux reconstruction**



## Admissible numerical solution

- Maximum principle / positivity preserving
- Prevent the code from crashing (for instance avoiding NaN)
- **Ensure the conservation of the scheme**

## Spurious oscillations

- Discrete maximum principle
- Relaxing condition for smooth extrema

## Accuracy

- Retain as much as possible the subcell resolution of the DG scheme
- Minimize the number of subcell solutions to recompute

**Modify locally, at the subcell level, the numerical solution without impacting the solution elsewhere in the cell**

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## DG as a subcell finite volume

- Rewrite DG scheme as a specific finite volume scheme on subcells
- Exhibit the corresponding subcell numerical fluxes: **reconstructed flux**

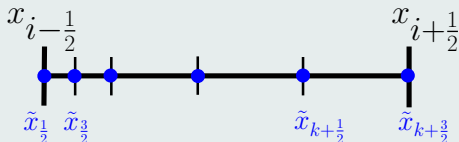
## Local variational formulation

$$\bullet \int_{\omega_i} \frac{\partial u_h^i}{\partial t} \psi \, dx = \int_{\omega_i} F(u_h^i) \frac{\partial \psi}{\partial x} \, dx - [\mathcal{F} \psi]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}, \quad \forall \psi \in \mathbb{P}^k(\omega_i)$$

- Substitute  $F(u_h^i)$  with  $F_h^i \in \mathbb{P}^{k+1}(\omega_i)$  (collocated or  $L_2$  projection)

$$\bullet \int_{\omega_i} \frac{\partial u_h^i}{\partial t} \psi \, dx = - \int_{\omega_i} \frac{\partial F_h^i}{\partial x} \psi \, dx + [(F_h^i - \mathcal{F}) \psi]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}, \quad \forall \psi \in \mathbb{P}^k(\omega_i)$$

## Subcell decomposition through $k + 2$ flux points



## Subdivision and definition

- $\omega_j$  is subdivided in  $k + 1$  subcells  $S_m^i = [\tilde{x}_{m-\frac{1}{2}}, \tilde{x}_{m+\frac{1}{2}}]$
- Let us define  $\bar{\psi}_m = \frac{1}{|S_m^i|} \int_{S_m^i} \psi \, dx$  the subcell mean value

## Subresolution basis functions

- Let us introduce the  $k + 1$  basis functions  $\{\phi_m\}_m$  such that  $\forall \psi \in \mathbb{P}^k(\omega_j)$

$$\int_{\omega_j} \phi_m \psi \, dx = \int_{S_m^i} \psi \, dx, \quad \forall m = 1, \dots, k + 1,$$

- $\sum_{m=1}^{k+1} \phi_m(x) = 1$

**These particular functions can be seen as the  $L_2$  projection of the indicator functions  $\mathbb{1}_m(x)$  onto  $\mathbb{P}^k(\omega_j)$**

## Subcell finite volume scheme

- $\int_{\omega_i} \frac{\partial u_h^i}{\partial t} \phi_m dx = - \int_{\omega_i} \frac{\partial F_h^i}{\partial x} \phi_m dx + \left[ (F_h^i - \mathcal{F}) \phi_m \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$
- $|S_m^i| \frac{\partial \bar{u}_m^i}{\partial t} = - \int_{S_m^i} \frac{\partial F_h^i}{\partial x} dx + \left[ (F_h^i - \mathcal{F}) \phi_m \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$
- $\frac{\partial \bar{u}_m^i}{\partial t} = - \frac{1}{|S_m^i|} \left( \left[ F_h^i \right]_{\tilde{x}_{m-\frac{1}{2}}}^{\tilde{x}_{m+\frac{1}{2}}} - \left[ \phi_m (F_h^i - \mathcal{F}) \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \right)$
- $\frac{\partial \bar{u}_m^i}{\partial t} = - \frac{1}{|S_m^i|} \left( \hat{F}_{m+\frac{1}{2}}^i - \hat{F}_{m-\frac{1}{2}}^i \right)$

Subcell finite volume

## Linear system

- $\hat{F}_{m+\frac{1}{2}}^i - \hat{F}_{m-\frac{1}{2}}^i = \left[ F_h^i \right]_{\tilde{x}_{m-\frac{1}{2}}}^{\tilde{x}_{m+\frac{1}{2}}} - \left[ \phi_m (F_h^i - \mathcal{F}) \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}, \quad \forall m \in \llbracket 1, k+1 \rrbracket$
- $\hat{F}_{\frac{1}{2}}^i = \mathcal{F}_{i-\frac{1}{2}} \quad \text{and} \quad \hat{F}_{k+\frac{3}{2}}^i = \mathcal{F}_{i+\frac{1}{2}}$

## Reconstructed flux

- $\widehat{F}_{m+\frac{1}{2}}^i = F_h^i(\widetilde{x}_{m+\frac{1}{2}}) - C_{m+\frac{1}{2}}^{i-\frac{1}{2}} \left( F_h^i(x_{i-\frac{1}{2}}) - \mathcal{F}_{i-\frac{1}{2}} \right) - C_{m+\frac{1}{2}}^{i+\frac{1}{2}} \left( F_h^i(x_{i+\frac{1}{2}}) - \mathcal{F}_{i+\frac{1}{2}} \right)$
- $C_{m+\frac{1}{2}}^{i-\frac{1}{2}} = \sum_{p=m+1}^{k+1} \phi_p(x_{i-\frac{1}{2}})$       and       $C_{m+\frac{1}{2}}^{i+\frac{1}{2}} = \sum_{p=1}^m \phi_p(x_{i+\frac{1}{2}})$

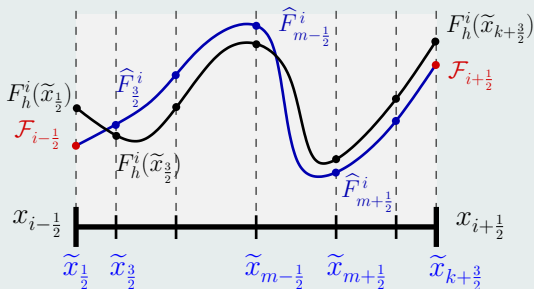
## Correction terms for symmetric distribution of $\{\widetilde{x}_{m+\frac{1}{2}}\}_m$

- Let  $\mathbf{B} \in \mathbb{R}^{k+1}$  be defined as  $B_j = (-1)^{j+1} \binom{k+j}{j} \binom{k+1}{j}$
- $\widetilde{\xi}_{m+\frac{1}{2}} = \frac{\widetilde{x}_{m+\frac{1}{2}} - x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}, \quad \forall m = 0, \dots, k+1$
- $C_{m+\frac{1}{2}}^{i-\frac{1}{2}} = 1 - \left( \widetilde{\xi}_{m+\frac{1}{2}}, \dots, (\widetilde{\xi}_{m+\frac{1}{2}})^{k+1} \right)^t \cdot \mathbf{B}$       and       $C_{m+\frac{1}{2}}^{i+\frac{1}{2}} = C_{k+\frac{3}{2}-m}^{i-\frac{1}{2}}$

## Subcell finite volume equivalent to DG

- $$\int_{\omega_i} \frac{\partial u_h^i}{\partial t} \psi \, dx = \int_{\omega_i} F(u_h^i) \frac{\partial \psi}{\partial X} \, dx - [\mathcal{F} \psi]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}, \quad \forall \psi \in \mathbb{P}^k(\omega_i)$$
- $$\frac{\partial \bar{u}_m^i}{\partial t} = -\frac{1}{|S_m^i|} \left( \widehat{F}_{m+\frac{1}{2}}^i - \widehat{F}_{m-\frac{1}{2}}^i \right), \quad \forall m = 1, \dots, k+1$$
- $$\widehat{F}_{m+\frac{1}{2}}^i = F_h^i(\tilde{x}_{m+\frac{1}{2}}) - C_{m+\frac{1}{2}}^{i-\frac{1}{2}} \left( F_h^i(x_{i-\frac{1}{2}}) - \mathcal{F}_{i-\frac{1}{2}} \right) - C_{m+\frac{1}{2}}^{i+\frac{1}{2}} \left( F_h^i(x_{i+\frac{1}{2}}) - \mathcal{F}_{i+\frac{1}{2}} \right)$$

## Reconstructed flux taking into account flux jumps



## Flux reconstruction / CPR

- The correction functions defined as

$$g_{LB}(x) = \sum_{m=0}^{k+1} C_{i-\frac{1}{2}}^{(m)} L_m(x) \quad \text{and} \quad g_{RB}(x) = \sum_{m=0}^{k+1} C_{i+\frac{1}{2}}^{(m)} L_m(x)$$

are nothing but the right and left Radau  $\mathbb{P}^k$  polynomials



H. T. HUYNH, *A Flux Reconstruction Approach to High-Order Schemes Including Discontinuous Galerkin Methods*. 18th AIAA Computational Fluid Dynamics Conference Miami, 2007.



Z.J. WANG and H. GAO, *A unifying lifting collocation penalty formulation including the discontinuous Galerkin, spectral volume/difference methods for conservation laws on mixed grids*. JCP, 2009.

## Subcell finite volume

- Reconstructed flux is used as a numerical flux for subcell FV schemes
- This demonstration is not restricted to the flux collocation case
- The correction terms are very simple and explicitly defined



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## RKDG scheme

- SSP Runge-Kutta: convex combinations of first-order forward Euler
- For sake of clarity, we focus on forward Euler time stepping

## Projection on subcells of RKDG solution

- $u_h^{i,n}(x) = \sum_{m=1}^{k+1} u_m^{i,n} \sigma_m(x)$  is uniquely defined by its  $k + 1$  submean values
- Introducing the matrix  $\mathbf{\Pi}$  defined as  $\pi_{mp} = \frac{1}{|S_m^i|} \int_{S_m^i} \sigma_p dx$ , then

$$\mathbf{\Pi} \begin{pmatrix} u_1^{i,n} \\ \vdots \\ u_{k+1}^{i,n} \end{pmatrix} = \begin{pmatrix} \bar{u}_1^{i,n} \\ \vdots \\ \bar{u}_{k+1}^{i,n} \end{pmatrix}$$

## Projection

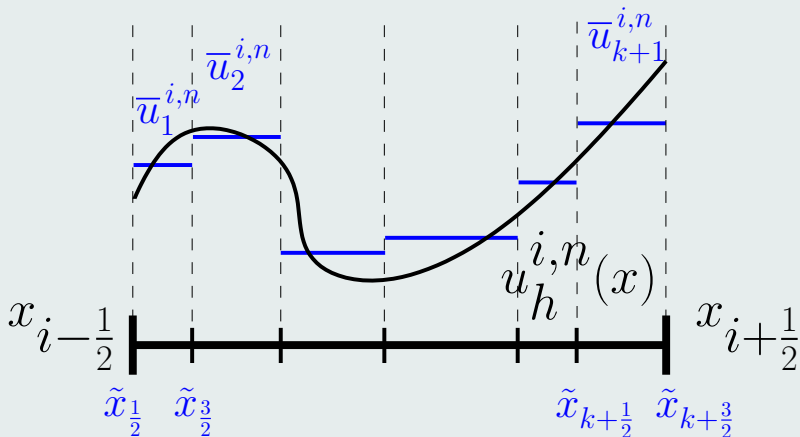


Figure : Polynomial solution and its associated submean values

## Set up

- We assume that, for each cell, the  $\{\bar{u}_m^{i,n}\}_m$  are admissible
- Compute a candidate solution  $u_h^{n+1}$  from  $u_h^n$  through uncorrected DG
- For each subcell, check if the submean values  $\{\bar{u}_m^{i,n+1}\}_m$  are ok

## Physical admissibility detection (PAD)

- Check if  $\bar{u}_m^{i,n+1}$  lies in an convex physical admissible set (maximum principle for SCL, positivity of the pressure and density for Euler, ...)
- Check if there is any NaN values

## Numerical admissibility detection (NAD)

- Discrete maximum principle DMP on submean values:

$$\min_p(\bar{u}_p^{i-1,n}, \bar{u}_p^{i,n}, \bar{u}_p^{i+1,n}) \leq \bar{u}_m^{i,n+1} \leq \max_p(\bar{u}_p^{i-1,n}, \bar{u}_p^{i,n}, \bar{u}_p^{i+1,n})$$

- This criterion needs to be relaxed to preserve smooth extrema

## Corrected reconstructed flux

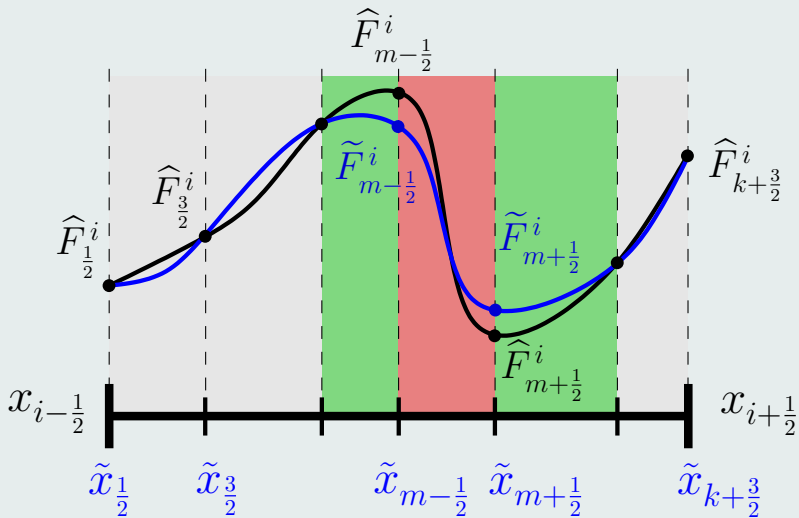


Figure : Correction of the reconstructed flux

## Flowchart

- 1 Compute the uncorrected DG candidate solution  $u_h^{i,n+1}$
- 2 Project  $u_h^{i,n+1}$  to get the submean values  $\bar{u}_m^{i,n+1}$
- 3 Check  $\bar{u}_m^{i,n+1}$  through the troubled zone detection plus relaxation
- 4 If  $\bar{u}_m^{i,n+1}$  is admissible go further in time, otherwise modify the corresponding reconstructed flux values

$$\tilde{F}_{m-1}^i = \mathcal{F}(\bar{u}_{m-1}^{i,n}, \bar{u}_m^{i,n}) \quad \text{and} \quad \tilde{F}_m^i = \mathcal{F}(\bar{u}_m^{i,n}, \bar{u}_{m+1}^{i,n})$$

- 5 Through the corrected reconstructed flux, recompute the submean values for tagged subcells and their first neighbors
- 6 Return to 3

## Conclusion

- The limitation only affects the DG solution at the subcell scale
- The corrected scheme is conservative at the subcell level
- In practice, few submean values need to be recomputed

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## Initial solution on $x \in [0, 1]$

- $u_0(x) = \sin(2\pi x)$
- Periodic boundary conditions

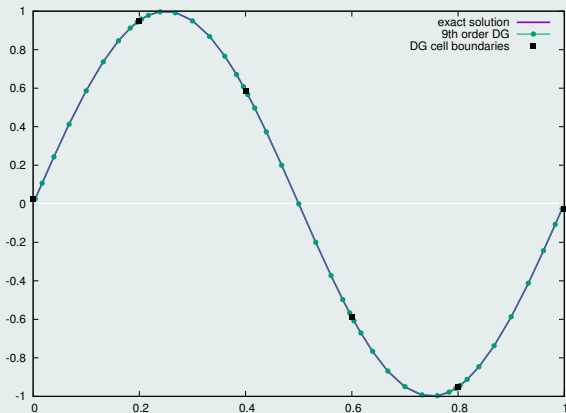


Figure : Linear advection with a 9th DG scheme and 5 cells after 1 period



## Convergence rates

	$L_1$		$L_2$	
$h$	$E_{L_1}^h$	$q_{L_1}^h$	$E_{L_2}^h$	$q_{L_2}^h$
$\frac{1}{20}$	8.07E-11	9.00	8.97E-11	9.00
$\frac{1}{40}$	1.58E-13	9.00	1.75E-13	9.00
$\frac{1}{80}$	3.08E-16	-	3.42E-16	-

**Table:** Convergence rates for the linear advection case for a 9th order DG scheme

# Linear advection of a square signal after 1 period

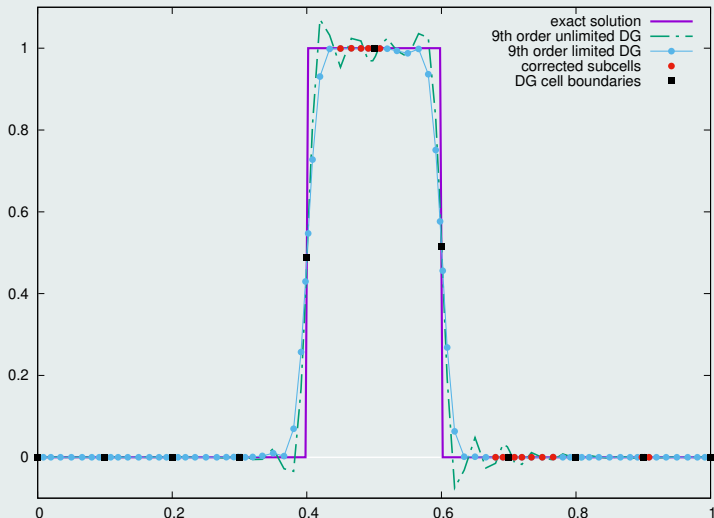


Figure : 9th order corrected and uncorrected DG solutions

# Linear advection of a square signal after 10 periods

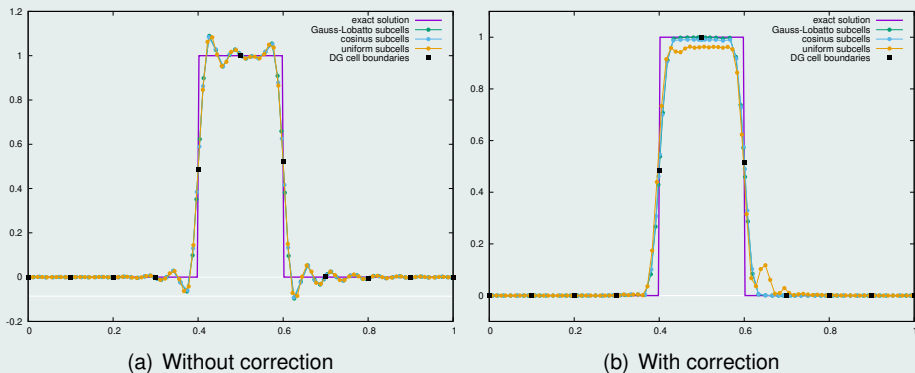
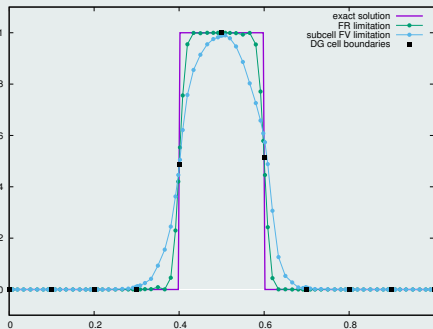
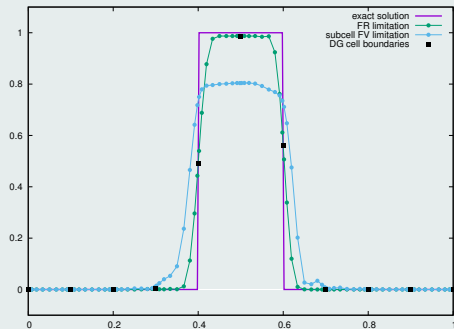


Figure : Comparison between different cell subdivision

# Linear advection of a square signal



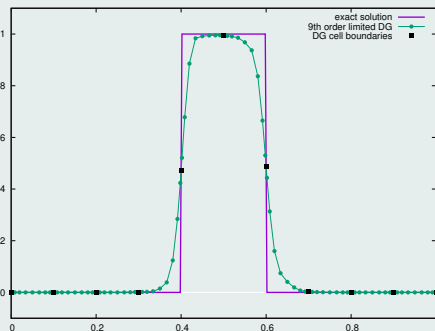
(a) After 1 period



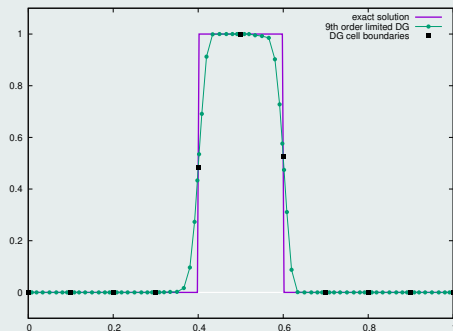
(b) After 50 periods

Figure : Comparison between subcell FV limitation and the present correction

# Linear advection of a square signal



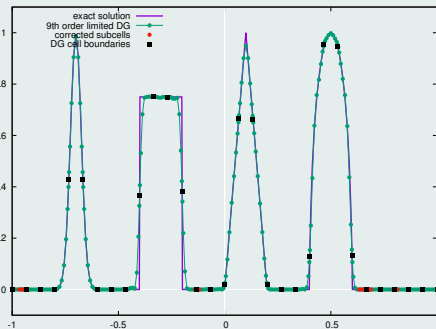
(a) 1st-order correction



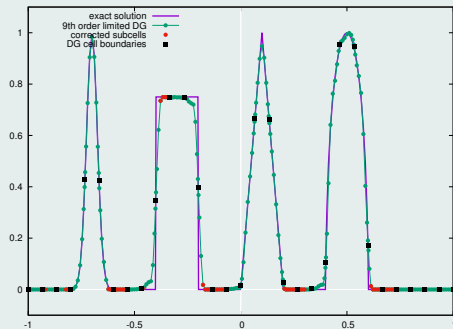
(b) 2nd-order correction

**Figure :** Comparison between 1st and 2nd order correction for the SubNAD detection criterion

# Linear advection of a composite signal after 4 periods



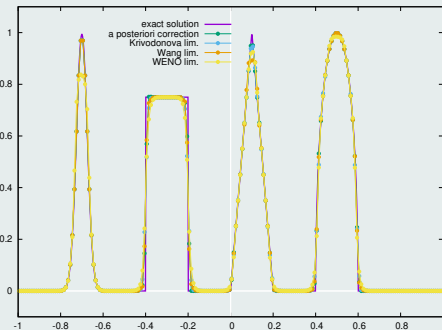
(a) NAD and 1st-order correction



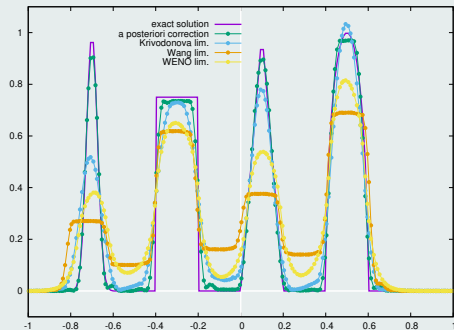
(b) SubNAD and 2nd-order correction

Figure : 9th order corrected DG on 30 cells

# Linear advection of a composite signal after 4 periods



(a) 200 cells: cell mean values



(b) 50 cells: subcell mean values

**Figure :** 4th order DG solutions provided different limitations

# Linear advection of a composite signal after 4 periods

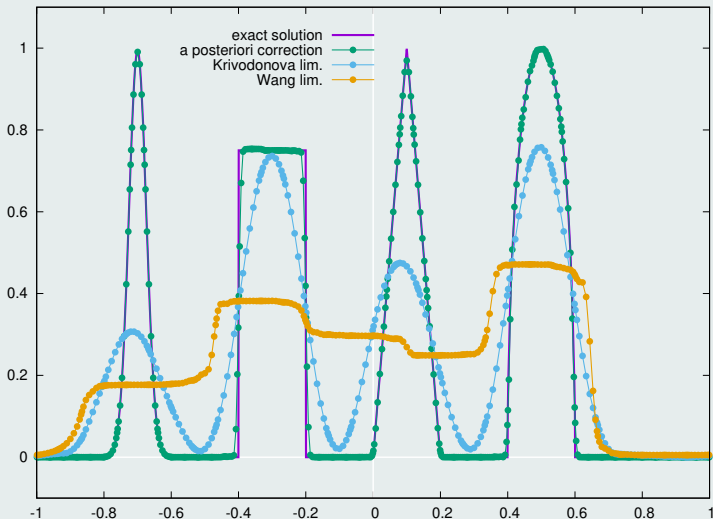


Figure : 9th order DG solutions provided different limitations on 30 cells



Burgers equation:  $u_0(x) = \sin(2\pi x)$

Figure : 9th order corrected DG on 10 cells for  $t_f = 0.7$

# Burgers equation: expansion and shock waves collision

Figure : 9th order corrected DG on 15 cells for  $t_f = 1.2$

# Burgers equation: expansion and shock waves collision

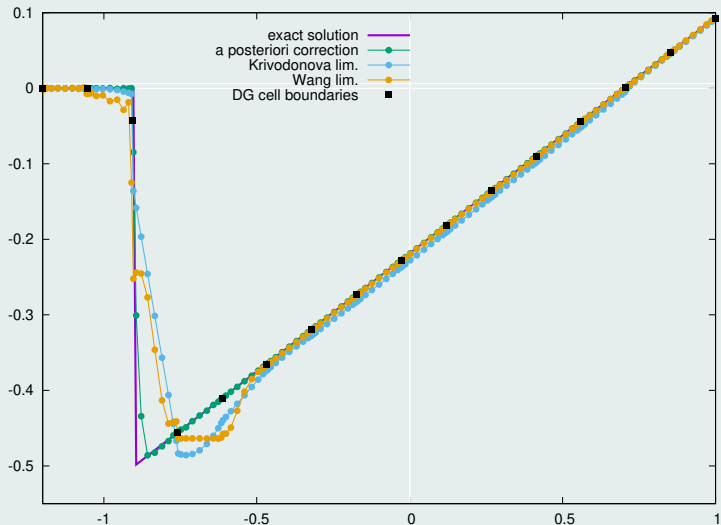
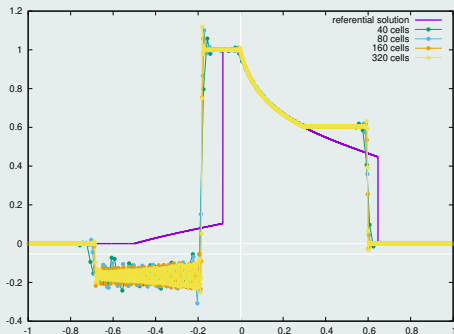
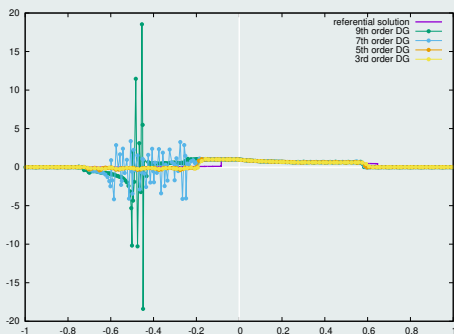


Figure : 9th order corrected DG on 15 cells provided different limitations

# Buckley non-convex flux problem at time $t = 0.4$



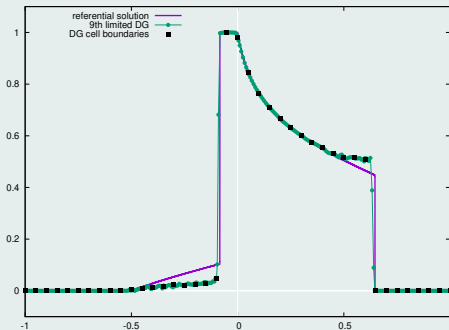
(a) Non-entropic behavior



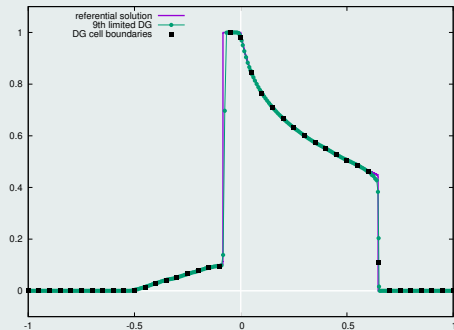
(b) Aliasing phenomenon

Figure : Uncorrected DG solution for the Buckley non-convex flux case

# Buckley non-convex flux problem at time $t = 0.4$



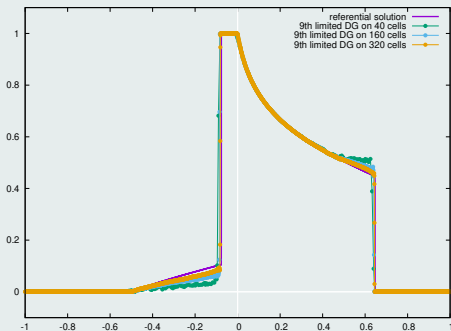
(a) NAD criterion



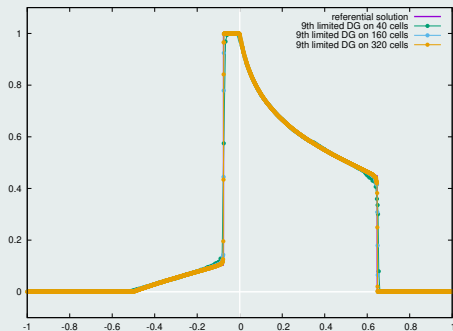
(b) SubNAD criterion

Figure : 9th order DG solutions on 40 cells

# Buckley non-convex flux problem at time $t = 0.4$



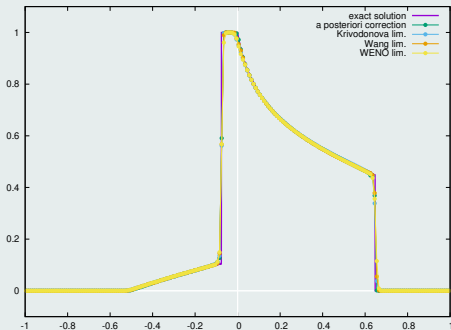
(a) NAD criterion



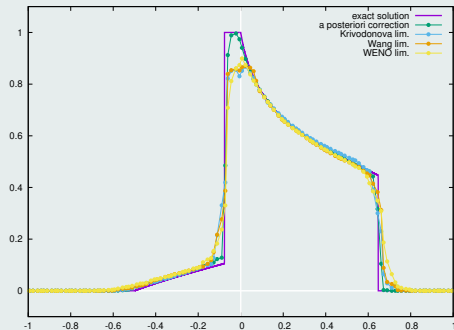
(b) SubNAD criterion

Figure : Convergence analysis of 9th order DG scheme

# Buckley non-convex flux problem at time $t = 0.4$



(a) 200 cells: cell mean values



(b) 30 cells: subcell mean values

Figure : 4th order DG solutions provided different limitations

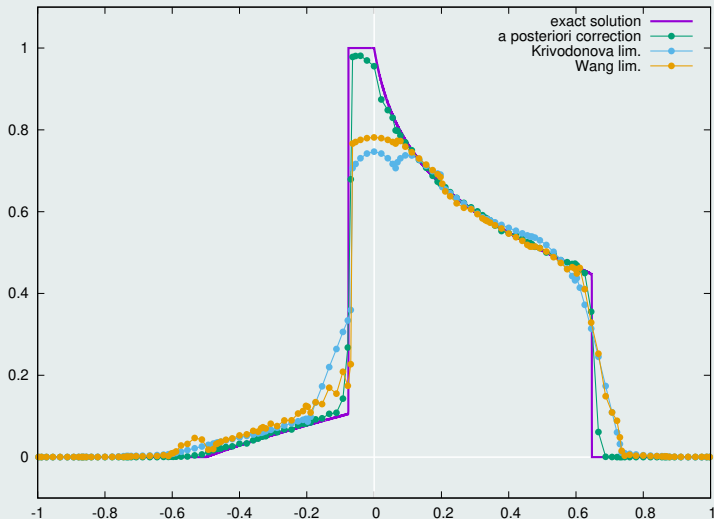
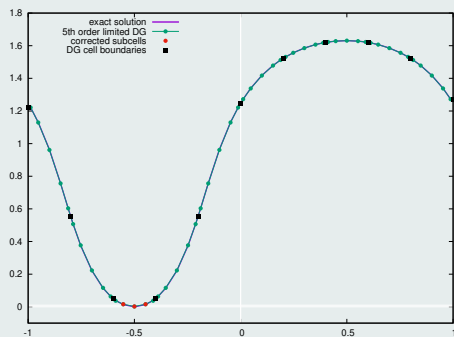
Buckley non-convex flux problem at time  $t = 0.4$ 

Figure : 9th order DG solutions provided different limitations on 15 cells

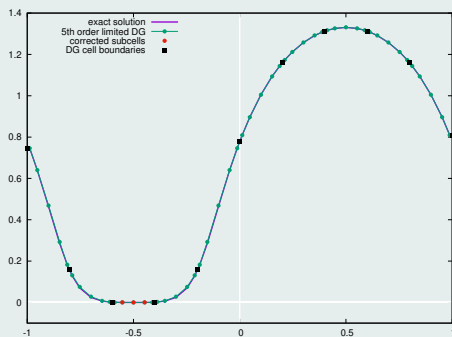


## Initial solution on $x \in [0, 1]$ for $\gamma = 3$

- $\rho_0(x) = 1 + 0.9999999 \sin(\pi x)$ ,  $u_0(x) = 0$ ,  $p_0(x) = (\rho_0(x))^\gamma$   
 $\implies \rho_0(-\frac{1}{2}) = 1.E - 7$  and  $p_0(-\frac{1}{2}) = 1.E - 21$
- Periodic boundary conditions



(a) Density



(b) Internal energy

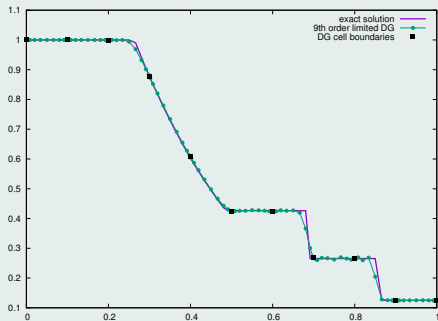
Figure : 5th order corrected DG solution on 10 cells at  $t = 0.1$

## Convergence rates

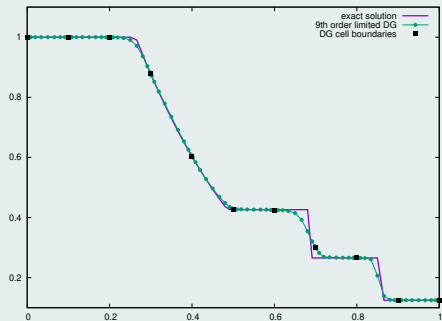
$h$	$L_1$		$L_2$		Average % of corrected subcells
	$E_{L_1}^h$	$q_{L_1}^h$	$E_{L_2}^h$	$q_{L_2}^h$	
$\frac{1}{20}$	1.48E-5	4.35	2.02E-5	4.18	6.87 %
$\frac{1}{40}$	9.09E-7	4.88	1.38E-6	4.87	3.31 %
$\frac{1}{80}$	3.09E-8	4.95	4.73E-8	4.86	2.50 %
$\frac{1}{160}$	1.00E-9	-	1.63E-9	-	1.12 %

Table: Convergence rates on the pressure for the Euler equation for a 5th order DG

# Sod shock tube problem



(a) NAD + 1st order correction



(b) SubNAD + 2nd order correction

Figure : 9th order corrected DG on 10 cells

# Sod shock tube problem

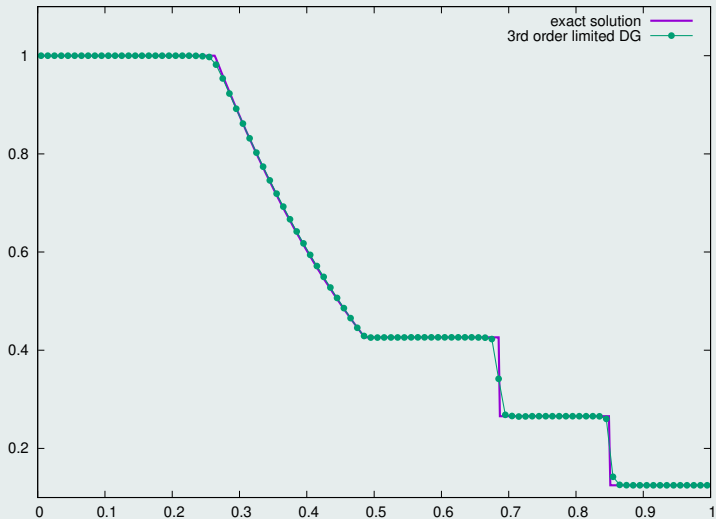
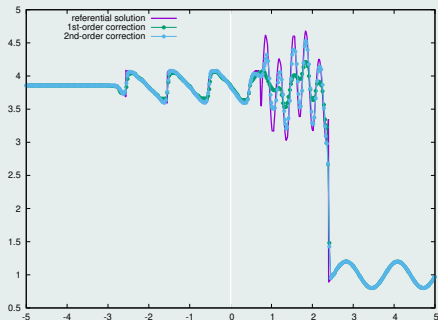
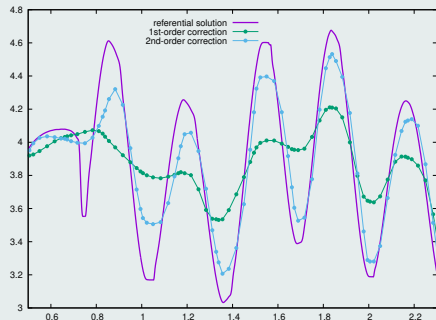


Figure : 3rd order DG solutions on 100 cells: cell mean values

# Shock acoustic-wave interaction problem



(a) Global view



(b) Zoom on  $[0.5, 2.3]$

**Figure :** 7th order corrected DG on 50 cells: comparison between 1st and 2nd order corrections

# Shock acoustic-wave interaction problem

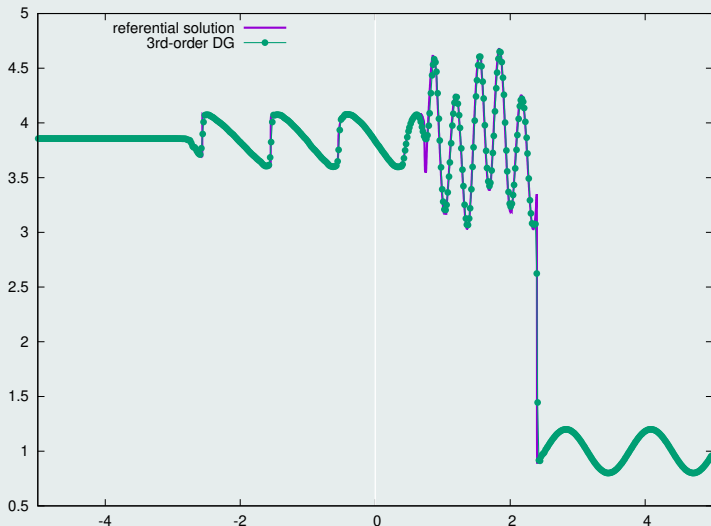


Figure : 3rd order corrected DG solutions on 200 cells: cell mean values

# Blast waves interaction problem

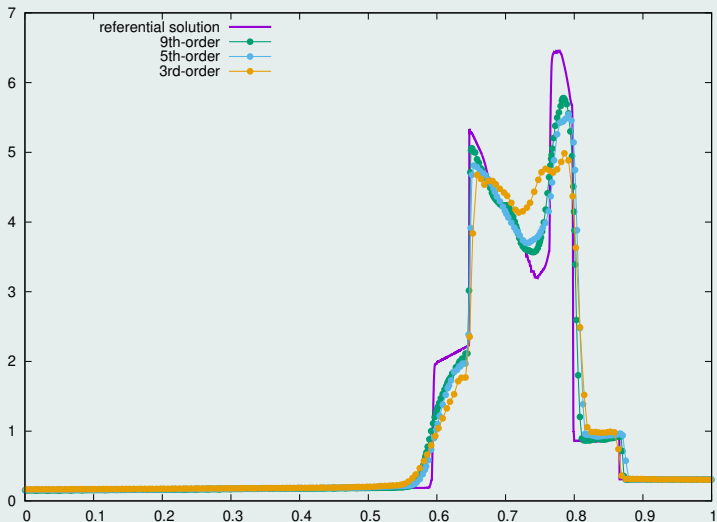
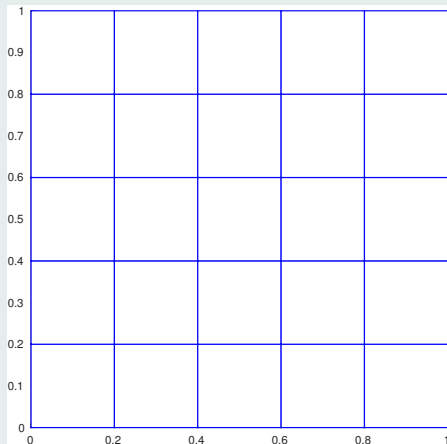
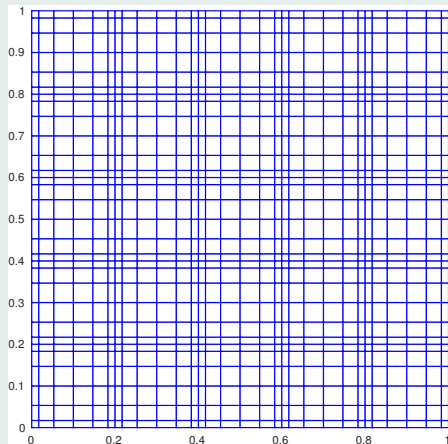


Figure : Corrected DG solution on 60 cells, from 3rd to 9th order

## 2D grid and subgrid



(a) Grid



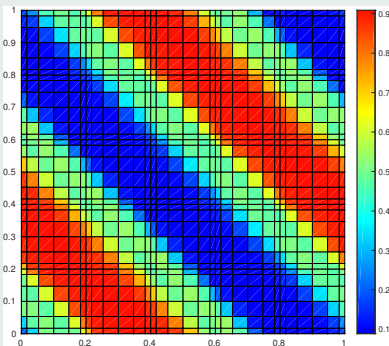
(b) Subgrid

**Figure :** 5x5 Cartesian grid and corresponding subgrid for a 6th order DG scheme

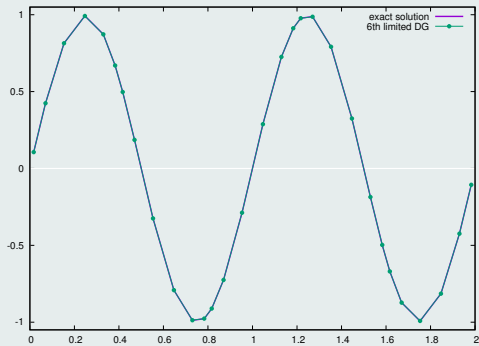


## Initial solution on $(x, y) \in [0, 1]^2$

- $u_0(x, y) = \sin(2\pi(x + y))$
- Periodic boundary conditions



(a) Solution map



(b) Solution profile

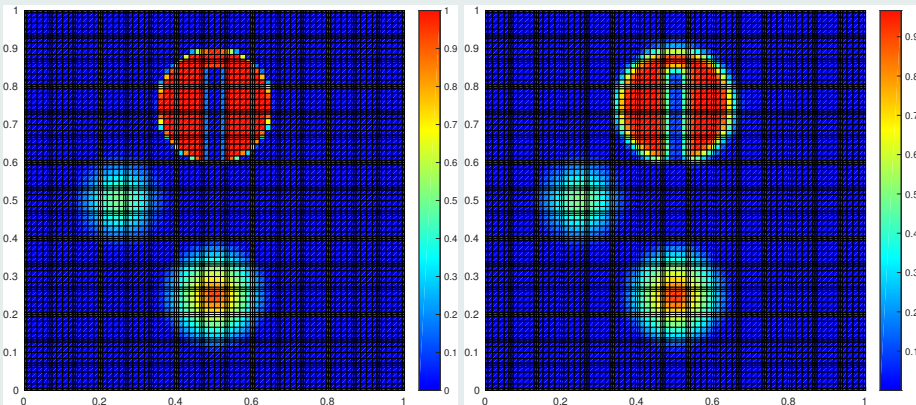
Figure : Linear advection with a 6th DG scheme and 5x5 grid after 1 period

## Convergence rates

	$L_1$		$L_2$	
$h$	$E_{L_1}^h$	$q_{L_1}^h$	$E_{L_2}^h$	$q_{L_2}^h$
$\frac{1}{5}$	2.10E-6	6.23	2.86E-6	6.24
$\frac{1}{10}$	2.79E-8	6.00	3.77E-8	6.00
$\frac{1}{20}$	3.36E-10	-	5.91E-10	-

**Table:** Convergence rates for the linear advection case for a 6th order DG scheme

# Rotation of a composite signal after 1 period

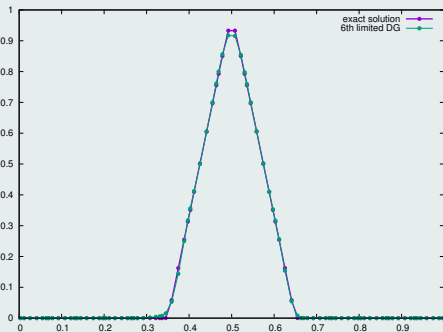


(a) Initial solution

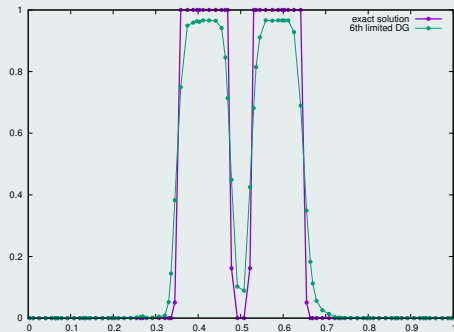
(b) Final solution

Figure : 6th order corrected DG on a 15x15 Cartesian mesh

## Rotation of a composite signal after 1 period



(a) Solution profile for  $y = 0.25$



(b) Solution profile for  $y = 0.75$

**Figure :** 6th order corrected DG on a 15x15 Cartesian mesh

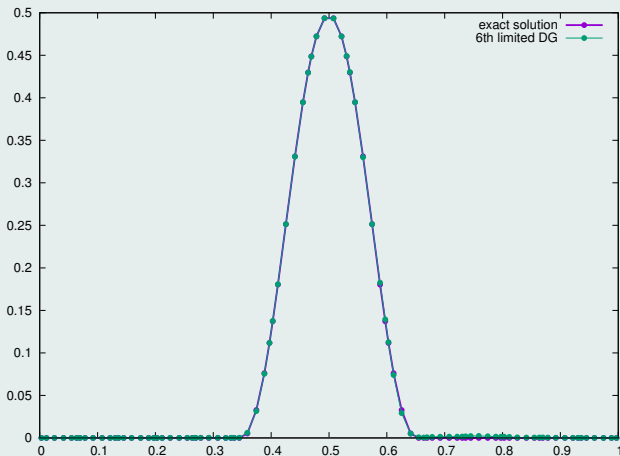
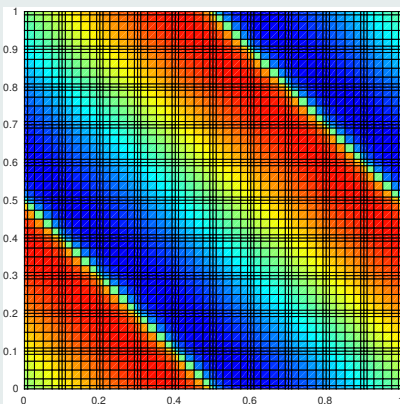
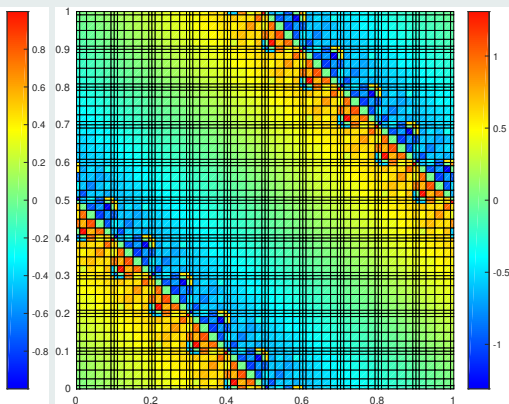
Rotation of a composite signal after 1 period:  $x = 0.25$ 

Figure : 6th order corrected DG on a 15x15 Cartesian mesh

# Burgers equation with $u_0(x, y) = \sin(2\pi(x + y))$



(a) Solution at  $t = 0.007$



(b) Solution at  $t = 0.25$

Figure : 6th order uncorrected DG on a  $10 \times 10$  Cartesian mesh

# Burgers equation with $u_0(x, y) = \sin(2\pi(x + y))$

(a) Solution map

(b) Detected subcells

Figure : 6th order corrected DG on a 10x10 Cartesian mesh until  $t = 0.5$

# Burgers equation with $u_0(x, y) = \sin(2\pi(x + y))$ at $t = 0.5$

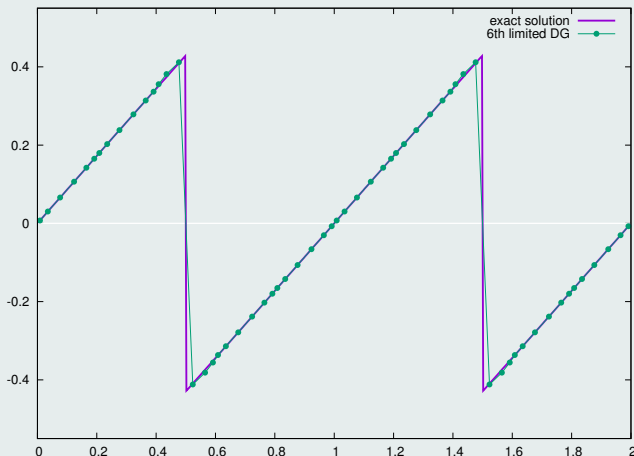
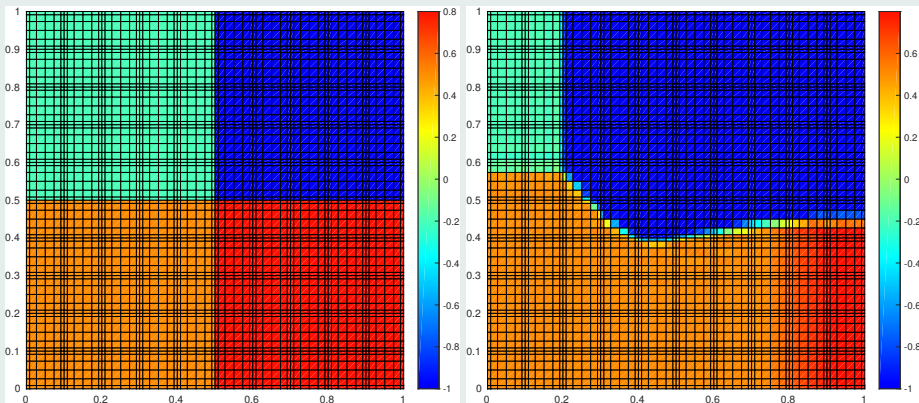


Figure : 6th order corrected DG solution profile on a 10x10 Cartesian mesh



# Burgers equation with composite signal



(a) Initial solution

(b) Solution at  $t = 0.5$

Figure : 6th order corrected DG on a 10x10 Cartesian mesh

# Kurganov, Petrova, Popov (KPP) non-convex flux problem

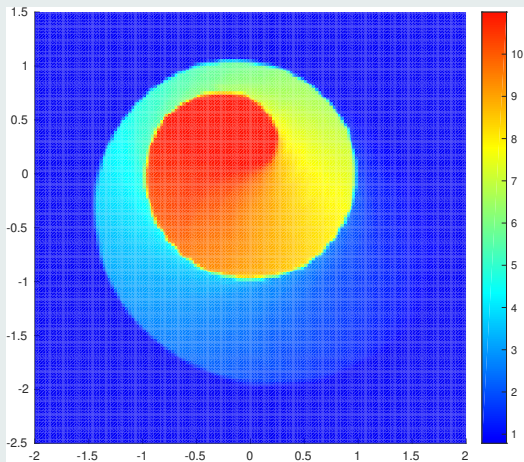


Figure : 6th order corrected DG solution on a 30x30 Cartesian mesh

## Ongoing work

- Extension to unstructured grids (with R. Abgrall)
- Maximum principle preserving DG scheme through subcell FCT reconstructed flux (with G. Gassner)
- DoF based  $h$ - $p$  adaptive DG scheme through subcell finite volume formulation (with R. Loubère and S. Clain)

## Published paper



F. VILAR, *A Posteriori Correction of High-Order Discontinuous Galerkin Scheme through Subcell Finite Volume Formulation and Flux Reconstruction*. JCP, DOI: 10.1016/j.jcp.2018.10.050, 2018.

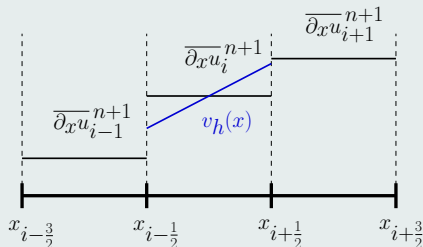
## Relaxation of the DMP

- $v_L = \overline{\partial_x u_i}^{n+1} - \frac{\Delta x_i}{2} \overline{\partial_{xx} u_i}^{n+1}$
- $v_{\min \setminus \max} = \min \setminus \max(\overline{\partial_x u_i}^{n+1}, \overline{\partial_x u_{i-1}}^{n+1})$
- If  $(v_L > \overline{\partial_x u_i}^{n+1})$  Then  $\alpha_L = \min(1, \frac{v_{\max} - \overline{\partial_x u_i}^{n+1}}{v_R - \overline{\partial_x u_i}^{n+1}})$
- If  $(v_L < \overline{\partial_x u_i}^{n+1})$  Then  $\alpha_L = \min(1, \frac{v_{\min} - \overline{\partial_x u_i}^{n+1}}{v_R - \overline{\partial_x u_i}^{n+1}})$
- $v_R = \overline{\partial_x u_i}^{n+1} + \frac{\Delta x_i}{2} \overline{\partial_{xx} u_i}^{n+1}$
- $v_{\min \setminus \max} = \min \setminus \max(\overline{\partial_x u_i}^{n+1}, \overline{\partial_x u_{i+1}}^{n+1})$
- If  $(v_R > \overline{\partial_x u_i}^{n+1})$  Then  $\alpha_R = \min(1, \frac{v_{\max} - \overline{\partial_x u_i}^{n+1}}{v_R - \overline{\partial_x u_i}^{n+1}})$
- If  $(v_R < \overline{\partial_x u_i}^{n+1})$  Then  $\alpha_R = \min(1, \frac{v_{\min} - \overline{\partial_x u_i}^{n+1}}{v_R - \overline{\partial_x u_i}^{n+1}})$

## Relaxation of the DMP

- $\alpha = \min(\alpha_L, \alpha_R)$
- If  $(\alpha = 1)$  Then DMP is relaxed

## Hierarchical limiter



- $v_h(x) = \overline{\partial_x u_i^{n+1}} + (x - x_i) \overline{\partial_{xx} u_i^{n+1}}$

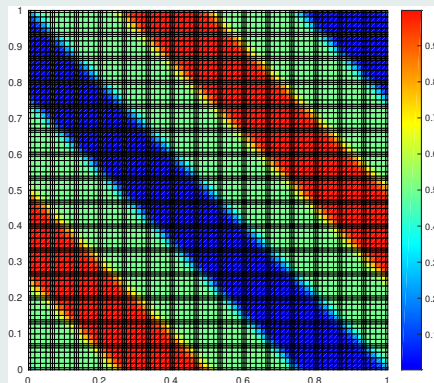


M. YANG and Z.J. WANG, *A parameter-free generalized moment limiter for high-order methods on unstructured grids*. AAMM., 2009.

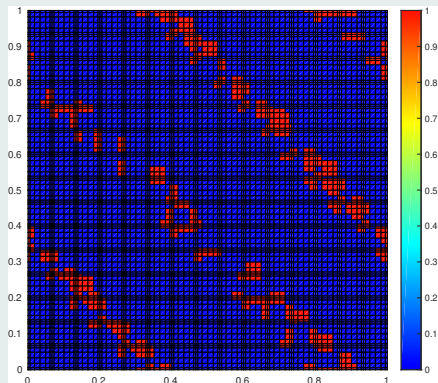


D. KUZMIN, *A vertex-based hierarchical slope limiter for p-adaptive discontinuous Galerkin methods*. J. of Comp. and Appl. Math., 2010.

## Linear advection of a square signal after 1 period



(a) Solution map



(b) Solution profile

Figure : 6th order corrected DG on a 15x15 Cartesian mesh

# Linear advection of a square signal after 1 period

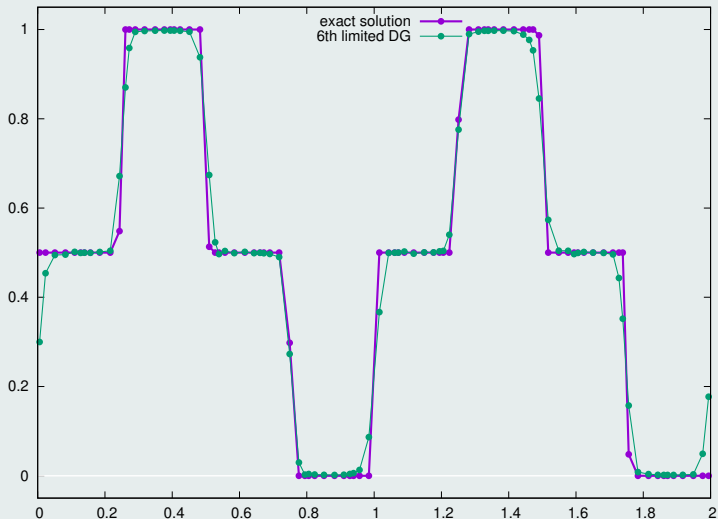


Figure : 6th order corrected DG on a 15x15 Cartesian mesh