A discontinuous Galerkin method for Lagrangian hydrodynamics

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1 Discontinuous Galerkin (DG) introduction with scalar conservation laws

1.1 Discretization

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0$$
$$u(x,0)=u^0(x)$$

goal: approximate our solution by polynomials on each cells without imposing continuity between them

• $\{e_j\}_{j=1..K}$ a basis of our approximation space $\mathbb{P}^K(C_i)$ and $u_h^i=\sum_{j=0}^{K}u_j^i(t)e_j^i(x)$ our approximate solution on C_i

$$\sum_{l=0}^{K}\partial_t u_l^i\int_{C_i}(e_l^i,e_k^i)dx+[\overline{f(u)}e_k^i]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}-\int_{C_i}f(u_h^i)\partial_x e_k^i dx=0$$

• $M_{kl}^i=\int_{C_i}(e_k^i,e_l^i)dx$, $D_{kl}^i=\int_{C_i}(\partial x e_k^i,e_l^i)dx$, $B^i(x)=(e_0^i(x),...,e_l^i(x),...,e_K^i(x))^T$, $F^i=(f_0^i,...,f_l^i,...,f_K^i)^T$, $U^i=(u_0^i,...,u_l^i,...,u_K^i)^T$ our unknown vector

$$M^i\frac{d}{dt}U^i+\left[\overline{f(u)}(x)B^i(x)\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}-D^iF^i=0$$

We could use local Taylor basis $\{e_j\}_{j=1..K}$ where $e_k^i=\frac{1}{k!}\left((\frac{x-x_i}{\Delta x_i})^k-\overline{(\frac{x-x_i}{\Delta x_i})^k}\right)$, x_i is the centroid of the cell C_i.

1.2 Numerical flux and L^2 stability

goal: access to the L^2 norm of our solution and insure stability

Mono-dimensional problems:

- f is integrable and its derivative smooth enough as $F(u)=\int_0^u f(s)ds$

$$\frac{d}{dt}\int_{C_i}\frac{u_h^2}{2}dx+[\overline{f(u)}u_h]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}-[F(u_h)]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}=0$$

- sum on all cells with $R_i=[\overline{f(u)}u_h]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}-[F(u_h)]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$
- permutation of the sum from cells to nodes and $u_g=u_h(x_{i+\frac{1}{2}}^-)$, $u_d=u_h(x_{i+\frac{1}{2}}^+)$

$\Rightarrow$ we find a sufficient condition on $f(u)$ with $C_{i+\frac{1}{2}}\geq 0$, to have $R_i\geq 0$:

$$\overline{f(u)}(x_{i+\frac{1}{2}})=\frac{1}{u_d-u_g}\int_{u_g}^{u_d}f(u)du-C_{i+\frac{1}{2}}(u_d-u_g)$$

- For linear advection, $\overline{f(u)}(x_{i+\frac{1}{2}})=\frac{a}{2}(u_g+u_d)-C_{i+\frac{1}{2}}(u_d-u_g)$:

1) $C_{i+\frac{1}{2}}=\frac{|a|}{2}$: upwind, 2) $C_{i+\frac{1}{2}}=\frac{\Delta x_i}{2\Delta t}$: Lax-Friedrichs, 3) $C_{i+\frac{1}{2}}=\frac{a^2}{2}\frac{\Delta t}{\Delta x_i}$: Lax-Wendroff

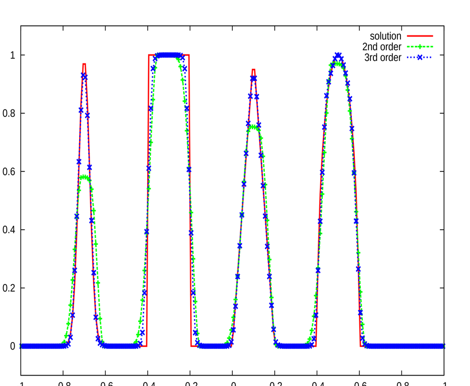
Multi-dimensional problems:

- same procedure and we find a similar expression for the numerical flux, on the face f_e, with $\overline{\overline{M_{f_e}}}$ a positive definite matrix:

$$\overline{f(u)}^{f_e}=\frac{1}{u_d-u_g}\int_{u_g}^{u_d}\overrightarrow{f(u)}du-(u_d-u_g)\overline{\overline{M_{f_e}}}\overrightarrow{n_{f_e}}$$

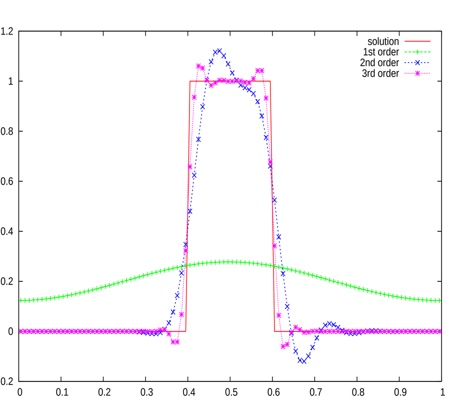
- Examples for linear advection: $\overline{\overline{M_{f_e}}}=\frac{1}{2}|\overrightarrow{A_{f_e}}.\overrightarrow{n_{f_e}}|I$ upwind scheme or $\overline{\overline{M_{f_e}}}=\frac{1}{2}|\overrightarrow{A_{f_e}}.\overrightarrow{n_{f_e}}|(\frac{\overrightarrow{A_{f_e}}\otimes\overrightarrow{A_{f_e}}}{||\overrightarrow{A_{f_e}}||^2})$

1.3 Limitation


Influence of the orders with limitation

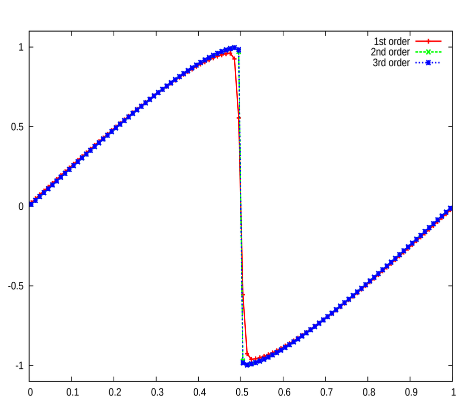
To enforce monotonicity, we perform a vertex based slope limitation [2].
2nd order, we have $u_h^i=u_0^i+\alpha_l u_1^i\frac{x-x_i}{\Delta x_i}$ with $\alpha_l\in[0,1]$, the correction factor
3rd order, we set $u_i^{(1)}=u_0^i+\alpha_l^{(1)}u_1^i\frac{x-x_i}{\Delta x_i}$ and $u_i^{(2)}=\frac{\partial}{\partial x}u_h^i=\frac{u_1^i}{\Delta x_i}+\alpha_l^{(2)}u_2^i\frac{x-x_i}{\Delta x_i^2}$.
In order to avoid the loss of accuracy at smooth extrema, we set $\alpha_l^{(1)}=\max(\alpha_l^{(1)},\alpha_l^{(2)})$. For high order, we calculate a nondecreasing sequence of correction factors $\alpha_l^{(p)}=\max(\alpha_l^{(q)})$, $q\geq p$, that means, as soon as $\alpha_l^{(q)}=1$ is encountered, we stop the limitation.

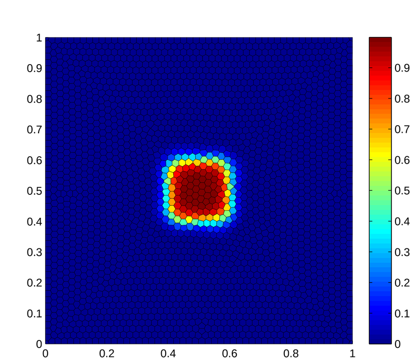
1.4 Numerical results

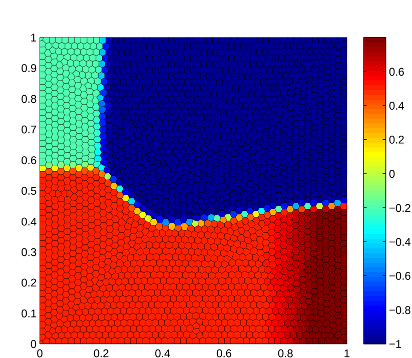

Influence of the orders on a linear advection problem

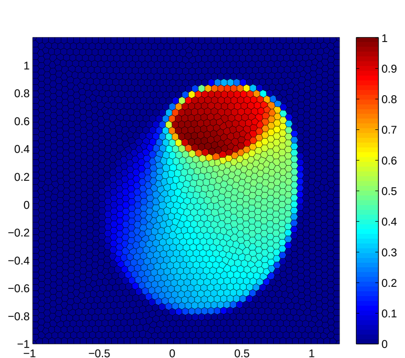
		L_1	L_2	L_∞
Linear advection	1st order	0.94	0.94	0.94
	2nd order	2.05	2.05	2.05
	2nd order lim	2.37	2.05	1.61
	3rd order	3.00	3.00	2.89
	3rd order lim	3.32	3.10	2.59
Burgers	1st order	0.86	0.68	0.23
	2nd order	2.00	1.99	1.91
	2nd order lim	2.12	1.99	1.57
	3rd order	2.88	2.91	2.65
	3rd order lim	2.87	2.89	2.62

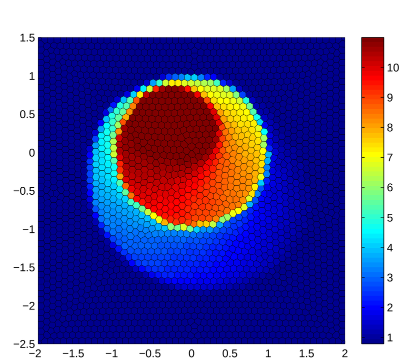
Numerical order of our methods in 1D


Influence of the orders on a Burgers problem


Linear advection problem with a 3rd order DG method on polygonal cells


Burgers problem with a 3rd order DG method on polygonal cells


Buckley problem with a 3rd order DG method on polygonal cells


KPP problem with a 3rd order DG method on polygonal cells

2 Lagrangian hydrodynamics

2.1 Gas dynamics in Lagrangian formalism

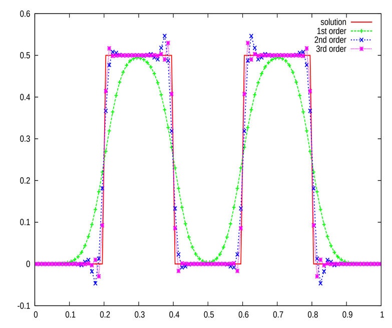
$$\rho^0\frac{d(1/\rho)}{dt}-\frac{\partial u}{\partial X}=0$$
$$\rho^0\frac{du}{dt}+\frac{\partial p}{\partial X}=0$$
$$\rho^0\frac{dE}{dt}+\frac{\partial pu}{\partial X}=0$$

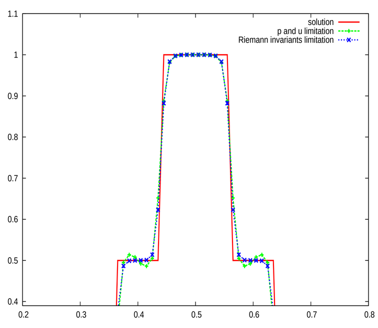
with ρ the density of the fluid, ρ^0 its initial density, u its velocity and E its total energy. For a thermodynamic closure of this system, we introduce an equation of state $p=p(\rho,\varepsilon)$ with $\varepsilon=E-\frac{u^2}{2}$.
We may, for example, use the ideal gas law : $p=\rho(\gamma-1)\varepsilon$.

2.2 Numerical results

2.2.1 Acoustic

- linearization by small perturbations of the gas dynamics system, around a steady flow $\Rightarrow (p,u)$ system

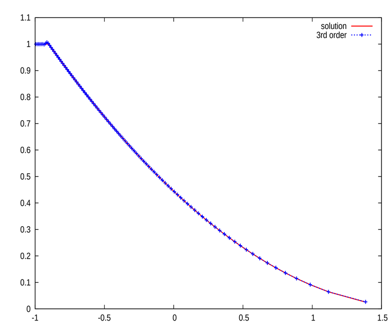

Influence of orders

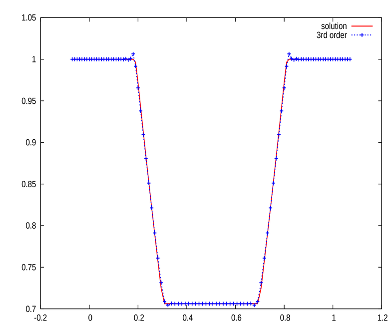

Influence of limitations

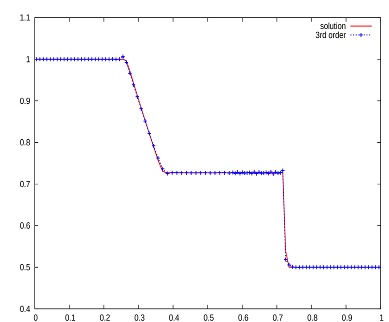
We notice that if we just perform the limitation on the system unknowns, some oscillations remain. But, by diagonalizing the system, we get around this constraint. For the acoustic system, it is quite simple because these invariants can be found explicitly (due to the linear property of this system) but for the other cases, it isn't so obvious.

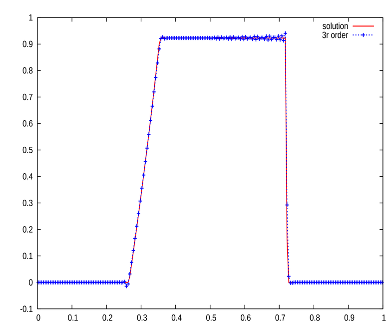
2.2.2 Shallow water

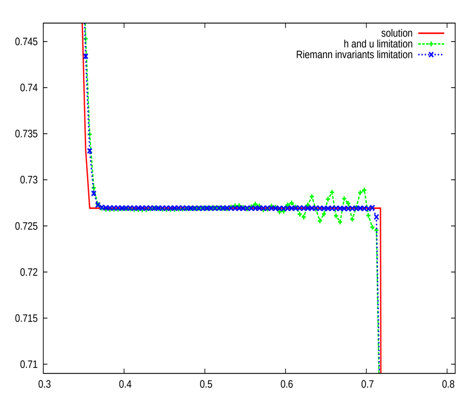
- small water height, an incompressible fluid, a sliding condition at the bottom and average of the equations on the water height $\Rightarrow (h,u)$ system where h is the water height


3rd order DG for a rarefaction wave into vacuum problem


3rd order DG for a double rarefaction waves problem


3rd order DG for a dam break problem: water height

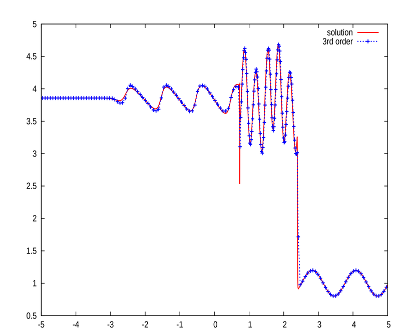

3rd order DG for a dam break problem: velocity

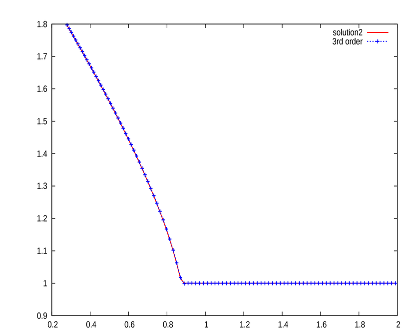


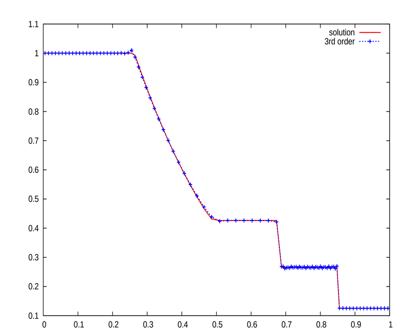
As before, if we apply our limitation on the intrinsic system variables, some oscillations remain. The problem is this equations system is nonlinear, and so we have only informations on the differential of the Riemann invariants. In order to access to this quantities, we've tested two different options :

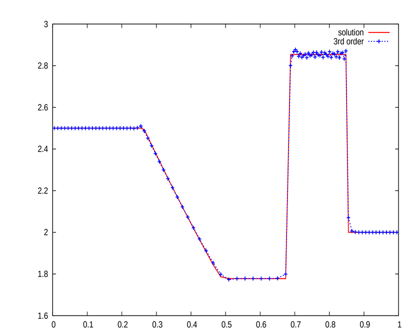
- these differentials being quite simple, we were able to integrate and differentiate them and so, perform our limitation on the Riemann invariants.
- we linearize, on each cells, the equations and thus, we obtain linear approximation of the Riemann invariants on each cells. Then, the limitation is easy.

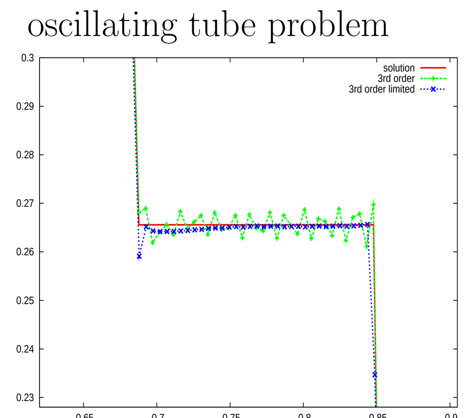
2.2.3 Gas dynamics


3rd order DG for a oscillating tube problem


3rd order DG for a uniformly accelerated piston problem


3rd order DG for a shock Sod tube problem: density


3rd order DG for a shock Sod tube problem: internal energy



We see that the oscillations are quite strong at the shock front, without any limitation. So, to keep our solution monotone, as we did for the shallow water equations, we linearize the system on each cells and obtain linear quantities on which we can perform our limitation. The problem is how can we limit our last unknown, E, the total energy. We have tested different ways but at the end, some little oscillations still remain.

3 Conclusions and perspectives

- Our DG methods have been validated and so, order influence on the accuracy was observed
- Different physical problems, linears and nonlinears, were studied in a Lagrangian formalism and explicit formulas for the flux, to have L_2 or entropy stability, have been shown
- Difficulties residing in limiting nonlinear systems have been noticed
- Multidimensional studies will be pursued for the Lagrangian hydrodynamics problem presented before
- Lagrangian schemes, using the initial mesh, will be studied, in order to avoid the cells deformation problem due to high orders

References

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