Subcell *a posteriori* limitation for DG scheme through flux reconstruction

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1D scalar conservation law

- $\dfrac{\partial u}{\partial t} + \dfrac{\partial F(u)}{\partial x} = 0, \qquad (x, t) \in \omega \times [0, T]$
- $u(x, 0) = u_0(x), \qquad x \in \omega$

$(k + 1)^{\text{th}}$ order discretization

- $\{\omega_i\}_i$ a partition of ω, such that $\omega_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$
- $0 = t^0 < t^1 < \cdots < t^N = T$ a partition of the temporal domain $[0, T]$
- $u_h(x, t)$ the numerical solution, such that $u_{h|\omega_i} = u_h^i \in \mathbb{P}^k(\omega_i)$

$$u_h^i(x, t) = \sum_{m=1}^{k+1} u_m^i(t)\, \sigma_m(x)$$

Local variational formulation on ω_i

- $\displaystyle \int_{\omega_i} \frac{\partial u_h^i}{\partial t} \psi \, dx = \int_{\omega_i} F(u_h^i) \frac{\partial \psi}{\partial x} \, dx - \left[\mathcal{F} \psi \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}, \qquad\qquad \forall \, \psi \in \mathbb{P}^k(\omega_i)$

- $\mathcal{F}_{i+\frac{1}{2}} = \mathcal{F}\left(u_h^i(x_{i+\frac{1}{2}}, t), u_h^{i+1}(x_{i+\frac{1}{2}}, t) \right)$ \hfill numerical flux
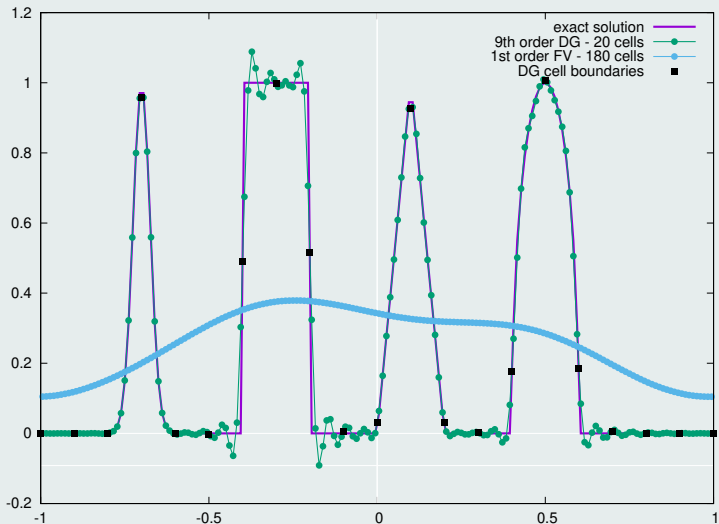
Subcell resolution of DG scheme

Figure : Linear advection of composite signal after 4 periods

Gibbs phenomenon

- High-order schemes leads to spurious oscillations near discontinuities
- Leads potentially to nonlinear instability, non-admissible solution, crash
- Vast literature of how prevent this phenomenon to happen:

  $\implies$ *a priori* and **a posteriori** limitations

A priori limitation

- Artificial viscosity
- Flux limitation
- Slope/moment limiter
- Hierarchical limiter
- ENO/WENO limiter

A posteriori limitation

- MOOD ("Multi-dimensional Optimal Order Detection")
- Subcell finite volume limitation
- **Subcell limitation through flux reconstruction**

Admissible numerical solution

- Maximum principle / positivity preserving
- Prevent the code from crashing (for instance avoiding NaN)
- **Ensure the conservation of the scheme**

Spurious oscillations

- Discrete maximum principle
- Relaxing condition for smooth extrema

Accuracy

- Retain as much as possible the subcell resolution of the DG scheme
- Minimize the number of subcell solutions to recompute

DG as a subcell finite volume

- Rewrite DG scheme as a specific finite volume scheme on subcells
- Exhibit the corresponding subcell numerical fluxes: **reconstructed flux**
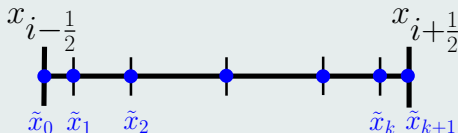
Local variational formulation

- $\displaystyle \int_{\omega_i} \frac{\partial u_h^i}{\partial t} \psi \, dx = \int_{\omega_i} F(u_h^i) \frac{\partial \psi}{\partial x} \, dx - \left[\mathcal{F} \psi \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}, \qquad \forall \, \psi \in \mathbb{P}^k(\omega_i)$

- Substitute $F(u_h^i)$ with $F_h^i \in \mathbb{P}^{k+1}(\omega_i)$ (collocated or L_2 projection)

- $\displaystyle \int_{\omega_i} \frac{\partial u_h^i}{\partial t} \psi \, dx = - \int_{\omega_i} \frac{\partial F_h^i}{\partial x} \psi \, dx + \left[(F_h^i - \mathcal{F}) \psi \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}, \qquad \forall \, \psi \in \mathbb{P}^k(\omega_i)$

Subcell decomposition through $k + 2$ flux points

Subresolution basis functions

- ω_i is subdivided in $k+1$ subcells $S_m^i = [\widetilde{x}_{m-1}, \widetilde{x}_m]$
- Let us introduce the $k+1$ basis functions $\{\phi_m\}_m$ such that $\forall \psi \in \mathbb{P}^k(\omega_i)$

$$\int_{\omega_i} \phi_m \, \psi \, \mathrm{d}x = \int_{S_m^i} \psi \, \mathrm{d}x, \qquad \forall \, m = 1, \ldots, k+1$$

- Let us define $\overline{\psi}_m = \dfrac{1}{|S_m^i|} \displaystyle\int_{S_m^i} \psi \, \mathrm{d}x$ the subcell mean value

Local variational formulation

- $\displaystyle\int_{\omega_i} \frac{\partial u_h^i}{\partial t} \phi_m \, \mathrm{d}x = -\int_{\omega_i} \frac{\partial F_h^i}{\partial x} \phi_m \, \mathrm{d}x + \left[(F_h^i - \mathcal{F}) \, \phi_m \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$

- $\dfrac{\partial \overline{u}_m^i}{\partial t} = -\dfrac{1}{|S_m^i|} \left(\left[F_h^i \right]_{\widetilde{x}_{m-1}}^{\widetilde{x}_m} - \left[\phi_m \left(F_h^i - \mathcal{F} \right) \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \right)$

- $\dfrac{\partial \overline{u}_m^i}{\partial t} = -\dfrac{1}{|S_m^i|} \left(\widehat{F}_m^i - \widehat{F}_{m-1}^i \right)$ Subcell finite volume

Linear system

- $\widehat{F}^i_m - \widehat{F}^i_{m-1} = \left[F^i_h \right]^{\widetilde{x}_m}_{\widetilde{x}_{m-1}} - \left[\phi_m \left(F^i_h - \mathcal{F} \right) \right]^{x_{i+\frac{1}{2}}}_{x_{i-\frac{1}{2}}}, \qquad \forall m = 1, \ldots, k$

- $\widehat{F}^i_0 = \mathcal{F}_{i-\frac{1}{2}} \qquad \text{and} \qquad \widehat{F}^i_{k+1} = \mathcal{F}_{i+\frac{1}{2}}$

Reconstructed flux

- $\widehat{F}^i_m = F^i_h(\widetilde{x}_m) - C^{(m)}_{i-\frac{1}{2}} \left(F^i_h(x_{i-\frac{1}{2}}) - \mathcal{F}_{i-\frac{1}{2}} \right) - C^{(m)}_{i+\frac{1}{2}} \left(F^i_h(x_{i+\frac{1}{2}}) - \mathcal{F}_{i+\frac{1}{2}} \right)$

- $C^{(m)}_{i-\frac{1}{2}} = \sum\limits_{p=m+1}^{k+1} \phi_p(x_{i-\frac{1}{2}}) \qquad \text{and} \qquad C^{(m)}_{i+\frac{1}{2}} = \sum\limits_{p=1}^{m} \phi_p(x_{i+\frac{1}{2}})$

Correction terms for symmetric distribution of $\{\widetilde{x}_m\}_m$

- Let $\boldsymbol{B} \in \mathbb{R}^{k+1}$ be defined as $\quad B_j = (-1)^{j+1} \dfrac{(k+1)(k+j)!}{(j!)^2(k+1-j)!}$

- $\widetilde{\xi}_m = \dfrac{\widetilde{x}_m - x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}, \quad \forall\, m = 0, \ldots, k+1$

- $C^{(m)}_{i-\frac{1}{2}} = 1 - \left(\widetilde{\xi}_m, \ldots, (\widetilde{\xi}_m)^{k+1} \right)^{\mathrm{t}} \cdot \boldsymbol{B} \qquad \text{and} \qquad C^{(m)}_{i+\frac{1}{2}} = C^{(k+1-m)}_{i-\frac{1}{2}}$

Subcell finite volume equivalent to DG

- $$\int_{\omega_i} \frac{\partial u_h^i}{\partial t}\, \psi \, \mathrm{d}x = \int_{\omega_i} F(u_h^i)\, \frac{\partial \psi}{\partial x}\, \mathrm{d}x - \left[\mathcal{F}\, \psi\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}, \qquad \forall\, \psi \in \mathbb{P}^k(\omega_i)$$

- $$\frac{\partial \overline{u}_m^i}{\partial t} = -\frac{1}{|S_m^i|}\left(\widehat{F}_m^i - \widehat{F}_{m-1}^i\right), \qquad \forall\, m = 1,\dots,k+1$$

- $$\widehat{F}_m^i = F_h^i(\widetilde{x}_m) - C_{i-\frac{1}{2}}^{(m)}\left(F_h^i(x_{i-\frac{1}{2}}) - \mathcal{F}_{i-\frac{1}{2}}\right) - C_{i+\frac{1}{2}}^{(m)}\left(F_h^i(x_{i+\frac{1}{2}}) - \mathcal{F}_{i+\frac{1}{2}}\right)$$

Reconstructed flux taking into account flux jumps
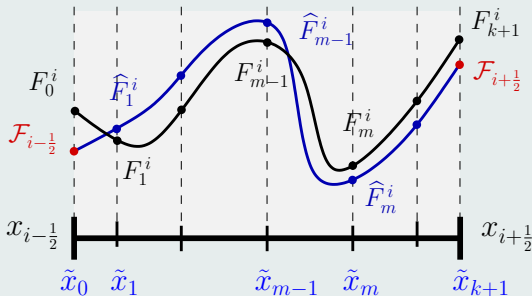
Flux reconstruction / CPR

- The correction functions defined as

$$g_{LB}(x) = \sum_{m=0}^{k+1} C_{i-\frac{1}{2}}^{(m)} L_m(x) \quad \text{and} \quad g_{RB}(x) = \sum_{m=0}^{k+1} C_{i+\frac{1}{2}}^{(m)} L_m(x)$$

  are nothing but the right and left Radau $\mathbb{P}^k$ polynomials

📄 H. T. HUYNH, *A Flux Reconstruction Approach to High-Order Schemes Including Discontinuous Galerkin Methods.* 18th AIAA Computational Fluid Dynamics Conference Miami, 2007.

📄 Z.J. WANG and H. GAO, *A unifying lifting collocation penalty formulation including the discontinuous Galerkin, spectral volume/difference methods for conservation laws on mixed grids.* JCP, 2009.

Subcell finite volume

- Reconstructed flux is used as a numerical flux for subcell FV schemes
- This demonstration is not restricted to the flux collocation case
- The correction terms are very simple and explicitly defined

Projection on subcells of RKDG solution

- $u_h^i = \sum_{m=1}^{k+1} u_m^i \sigma_m$ is uniquely defined by its $k+1$ submean values
- Introducing the matrix Π defined as $\pi_{mp} = \dfrac{1}{|S_m^i|} \displaystyle\int_{S_m^i} \sigma_p \, dx$, then

$$\Pi \left(u_1^{i,n}, \ldots, u_{k+1}^{i,n} \right)^t = \left(\overline{u}_1^{i,n}, \ldots, \overline{u}_{k+1}^{i,n} \right)^t$$

Polynomial solution and its associated submean values
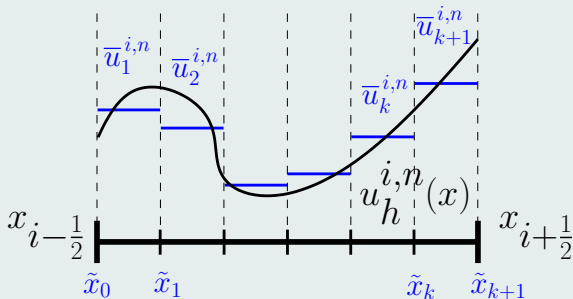
Set up

- We assume that, for each cell, the $\{\overline{u}_m^{i,n}\}_m$ are admissible
- Compute a candidate solution u_h^{n+1} from u_h^n through unlimited DG
- For each subcell, check if the submean values $\{\overline{u}_m^{i,n+1}\}_m$ are ok

Physical admissibility detection (PAD)

- Check if $\overline{u}_m^{i,n+1}$ lies in an convex physical admissible set (maximum principle for SCL, positivity of the pressure and density for Euler, ...)
- Check if there is any *NaN* values

Numerical admissibility detection (NAD)

- Discrete maximum principle DMP on submean values:

$$\min_p(\overline{u}_p^{i-1,n}, \overline{u}_p^{i,n}, \overline{u}_p^{i+1,n}) \leq \overline{u}_m^{i,n+1} \leq \max_p(\overline{u}_p^{i-1,n}, \overline{u}_p^{i,n}, \overline{u}_p^{i+1,n})$$

- This criterion needs to be relaxed to preserve smooth extrema

Limited reconstructed flux
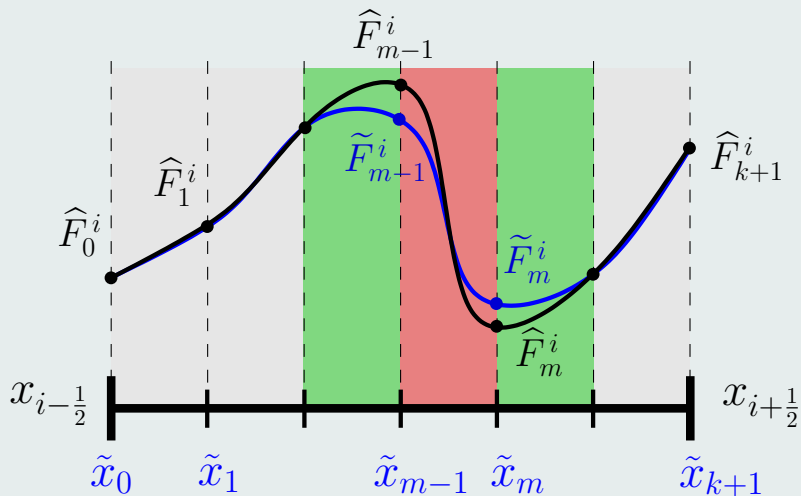
Figure : Correction of the reconstructed flux

Flowchart

1. Project $u_h^{i,n+1}$ to get the submean values $\overline{u}_m^{i,n+1}$

2. Check $\overline{u}_m^{i,n+1}$ through PAD and NAD

3. If $\overline{u}_m^{i,n+1}$ is admissible go further in time, otherwise modify the corresponding reconstructed flux values

$$\widetilde{F}_{m-1}^i = \mathcal{F}(\overline{u}_{m-1}^{i,n}, \overline{u}_m^{i,n}) \quad \text{and} \quad \widetilde{F}_m^i = \mathcal{F}(\overline{u}_m^{i,n}, \overline{u}_{m+1}^{i,n})$$

4. Through the corrected reconstructed flux, recompute the submean values for tagged subcells and their first neighbors

5. Return to point 2

Conclusion

- The limitation only affects the DG solution at the subcell scale
- The limited scheme is conservative at the subcell level
- In practice, few submean values need to be recomputed

Linear advection of a square signal after 1 period
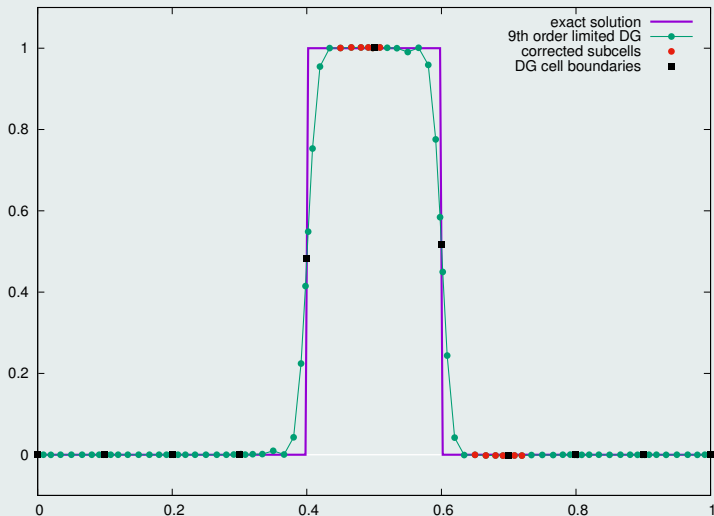
Figure : 9th order limited DG: NAD criterion

Linear advection of a square signal after 1 period
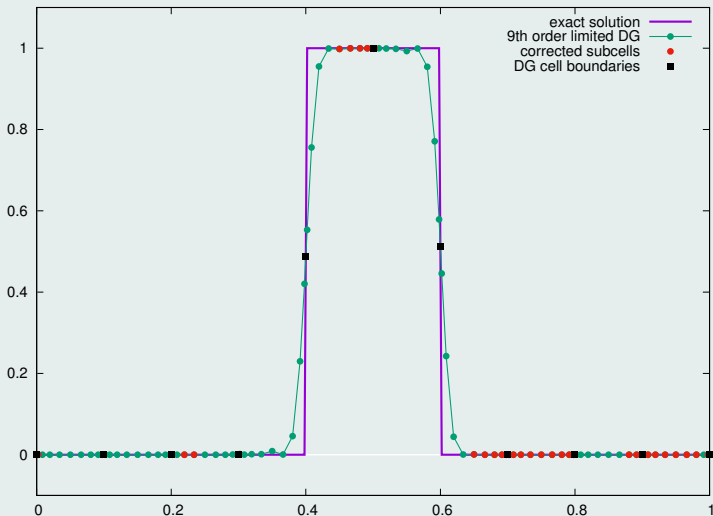
Figure : 9th order limited DG on 10 cells: NAD and PAD criteria

Linear advection of a square signal after 1 period
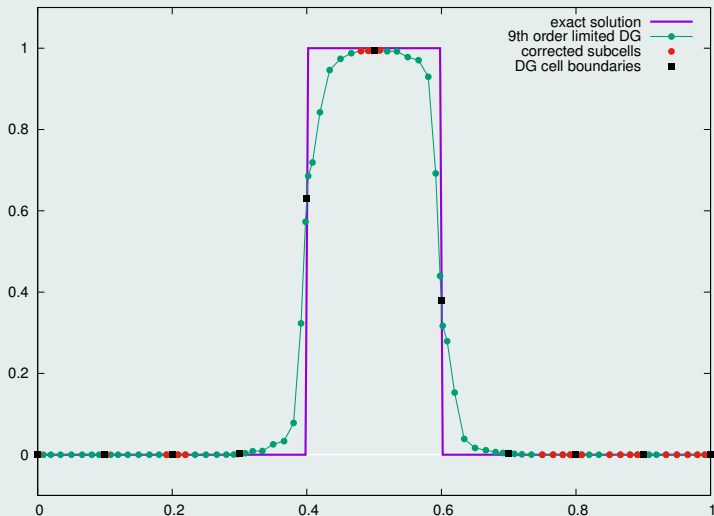
Figure : 9th order limited DG on 10 cells: subcell DMP

Linear advection of a square signal after 1 period
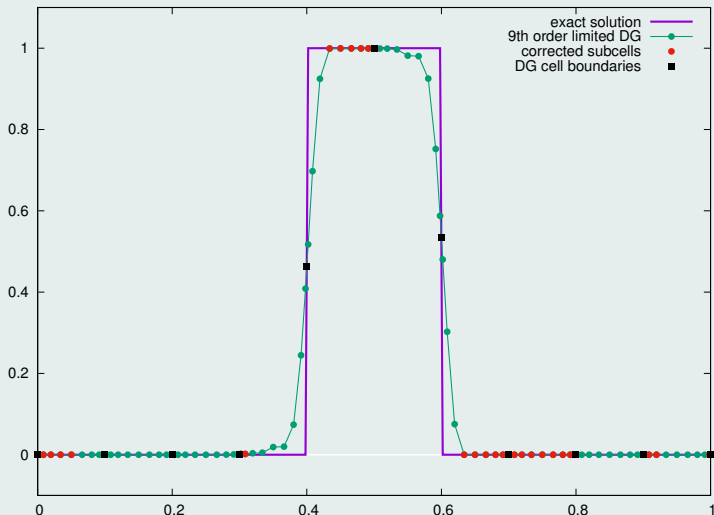
Figure : 9th order limited DG on 10 cells: subcell DMP + 2nd order correction

Linear advection of a composite signal after 4 periods
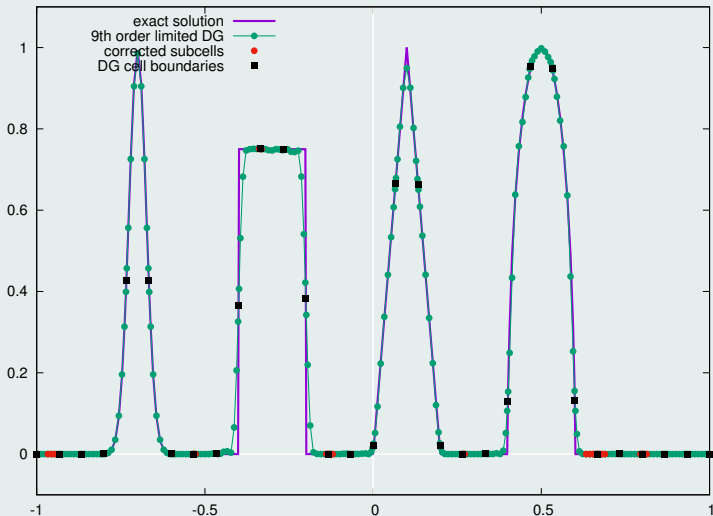
Figure : 9th order limited DG after 4 periods on 30 cells

Linear advection of a composite signal after 4 periods
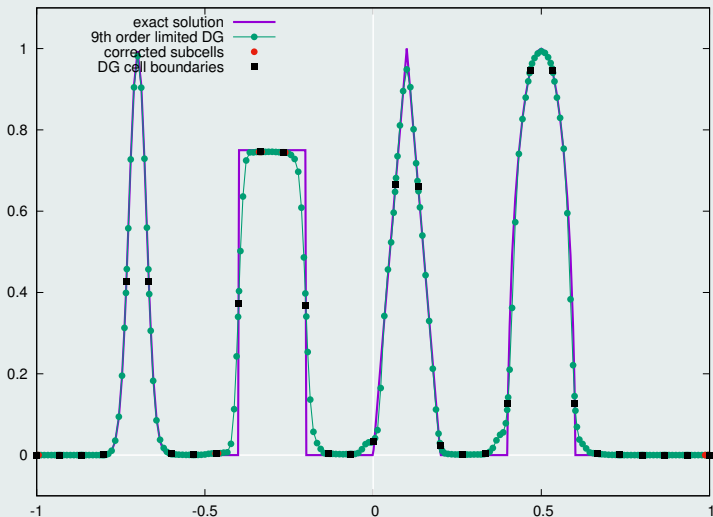
Figure : 9th order limited DG after 4 periods on 30 cells: subcell DMP

Linear advection of a composite signal after 4 periods
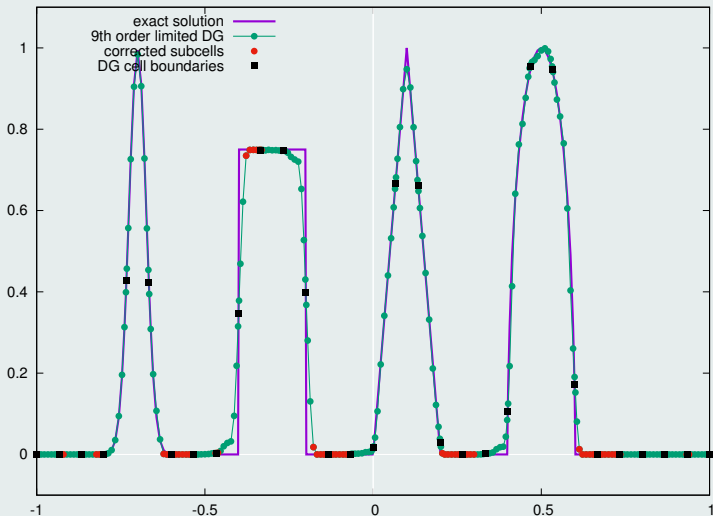
Figure : 9th order limited DG after 4 periods on 30 cells: subcell DMP + 2nd order correction

Burgers equation: $u_0(x) = \sin(2\pi x)$

Figure : 9th order limited DG on 10 cells for $t_f = 0.7$

Burgers equation: expansion and shock waves collision

Figure : 9th order limited DG on 15 cells for $t_f = 1.2$

2D grid and subgrid
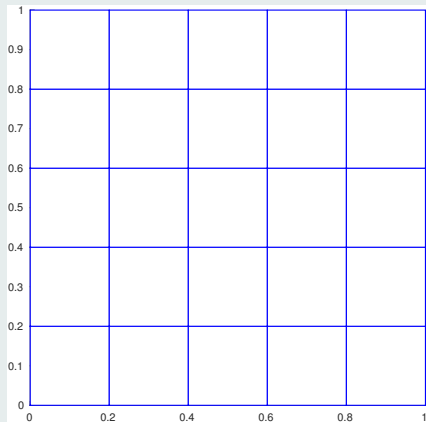
(a) Grid
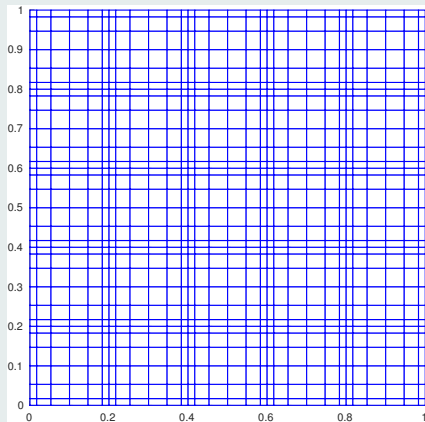
(b) Subgrid

Figure : 5x5 Cartesian grid and corresponding subgrid for a 6th order DG scheme

Linear advection of a square signal after 1 period
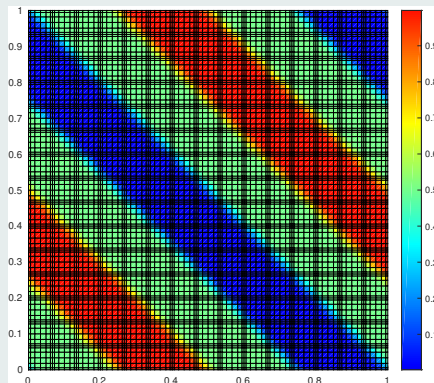
(c) Solution map
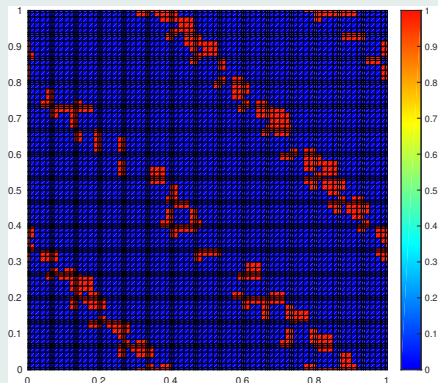
(d) Solution profile

Figure : 6th order limited DG on a 15x15 Cartesian mesh

Linear advection of a square signal after 1 period
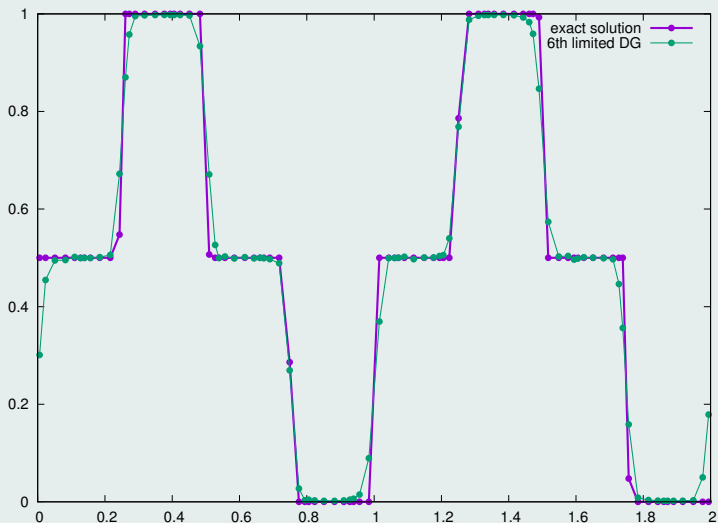
Figure : 6th order limited DG on a 15x15 Cartesian mesh

Rotation of a composite signal after 1 period
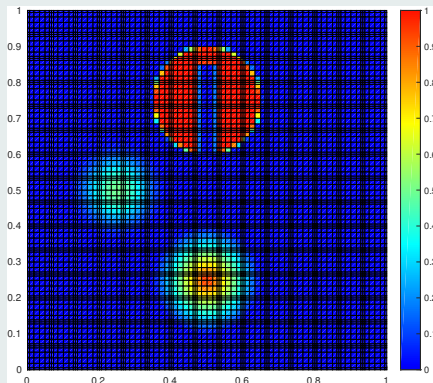
(e) Initial solution
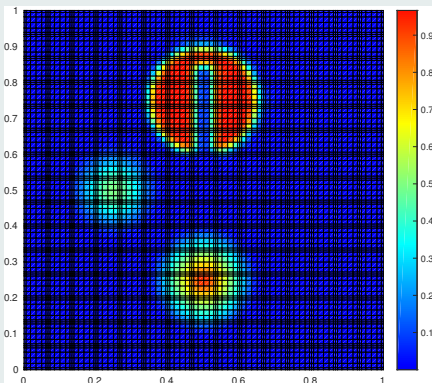
(f) Final solution

Figure : 6th order limited DG on a 15x15 Cartesian mesh

Rotation of a composite signal after 1 period
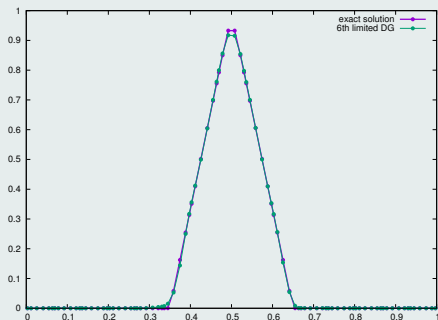
(g) Solution profile for $y = 0.25$
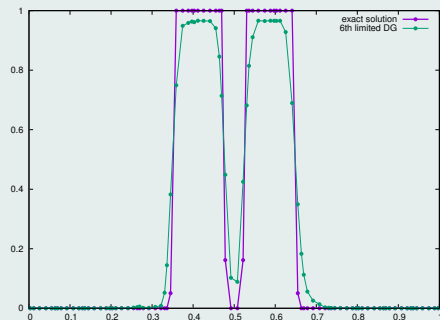
(h) Solution profile for $y = 0.75$

Figure : 6th order limited DG on a 15x15 Cartesian mesh

Rotation of a composite signal after 1 period: $x = 0.25$
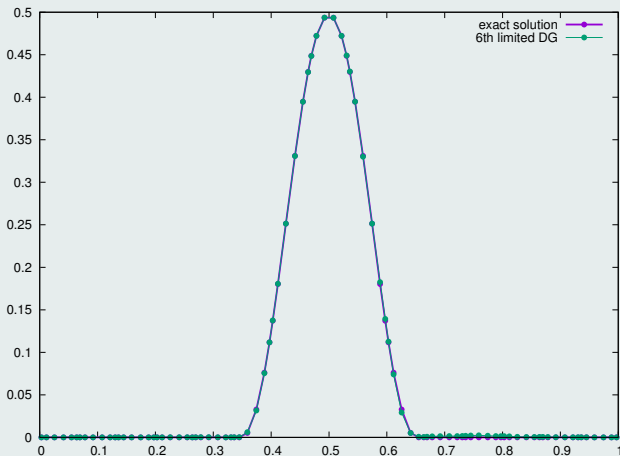
Figure : 6th order limited DG on a 15x15 Cartesian mesh

Burgers equation with $u_0(x, y) = \sin(2\pi (x + y))$
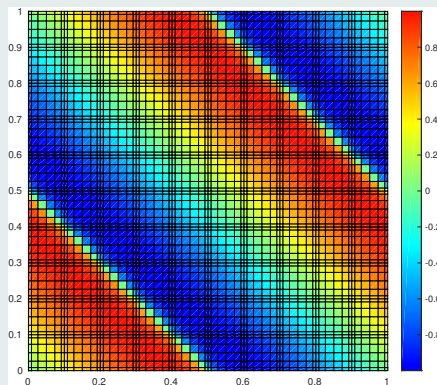
(i) Solution at $t = 0.007$
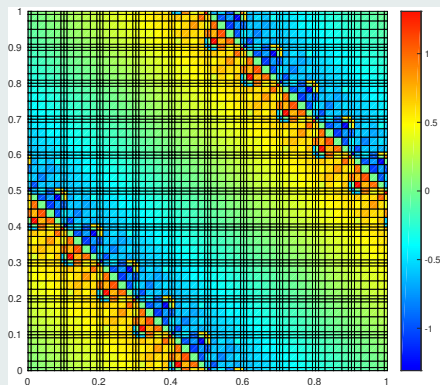
(j) Solution at $t = 0.25$

Figure : 6th order unlimited DG on a 10x10 Cartesian mesh

Burgers equation with $u_0(x, y) = \sin(2\pi (x + y))$

(k) Solution map   (l) Detected subcells

Figure : 6th order limited DG on a 10x10 Cartesian mesh until $t = 0.5$

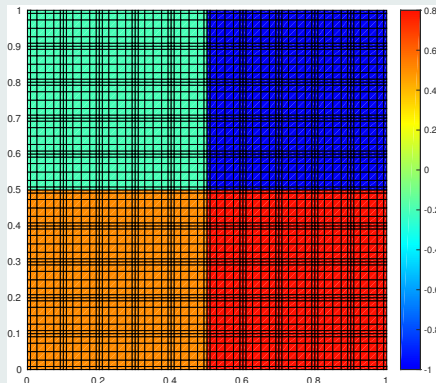# Burgers equation with $u_0(x, y) = \sin(2\pi (x + y))$ at $t = 0.5$
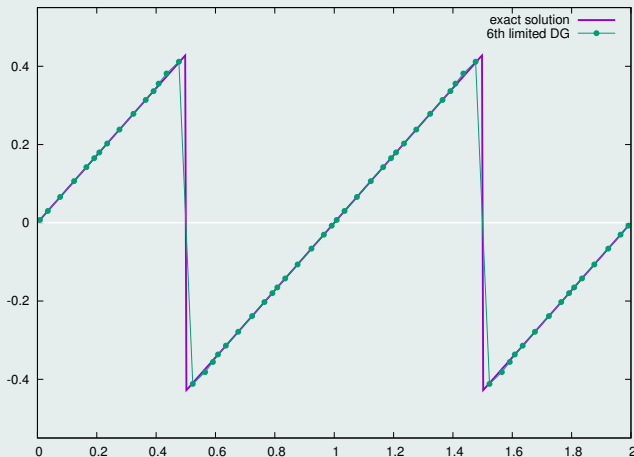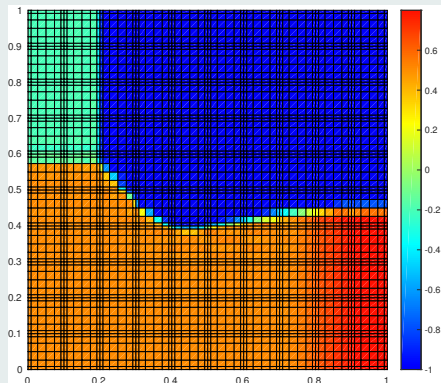

Figure : 6th order limited DG density profile on a 10x10 Cartesian mesh

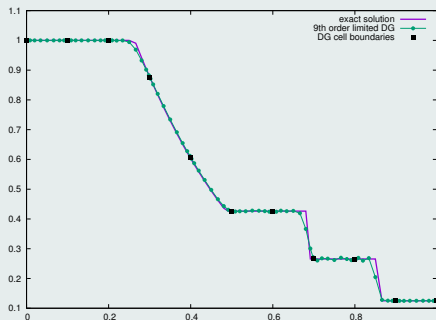# Burgers equation with composite signal
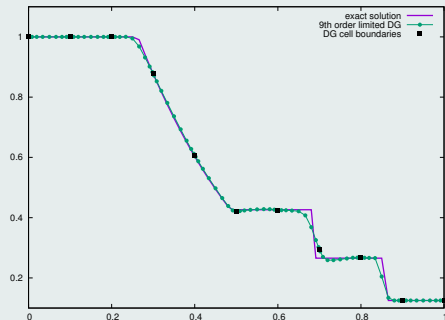

(m) Initial solution

(n) Solution at $t = 0.5$

Figure : 6th order limited DG on a 10x10 Cartesian mesh

Sod shock tube problem

(o) PAD + NAD

(p) PAD + subDMP with 2nd order correction

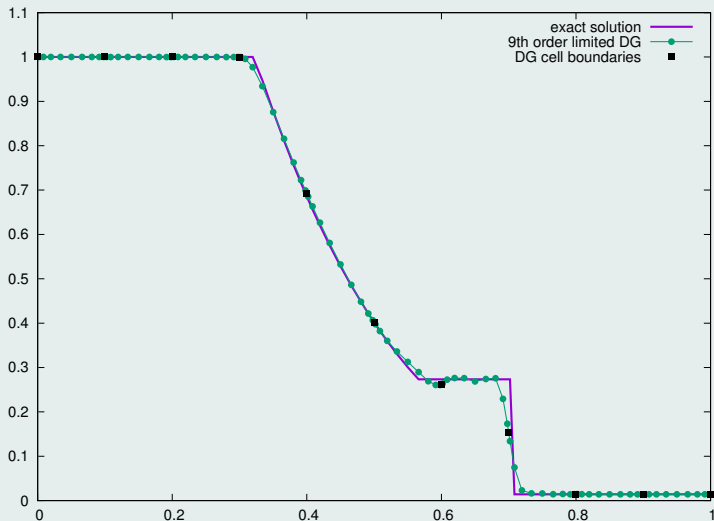Figure : 9th order limited DG on 10 cells

Hell shock tube problem

Figure : 9th order limited DG on 10 cells

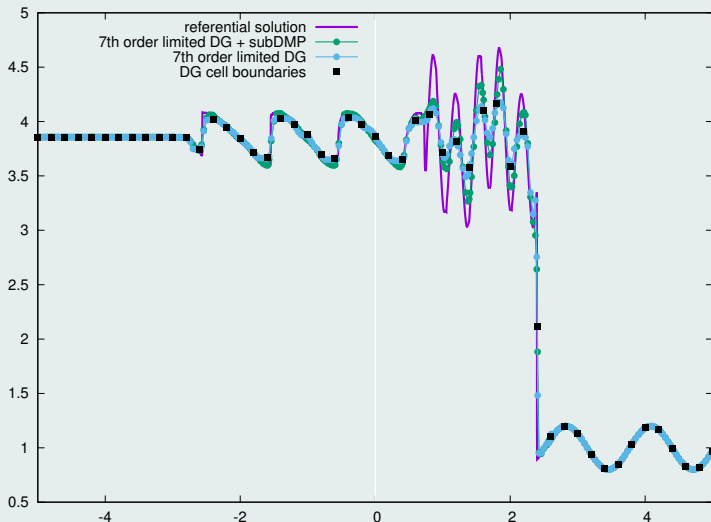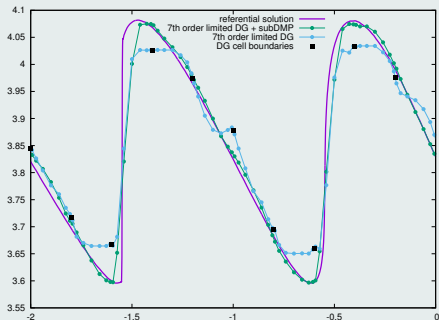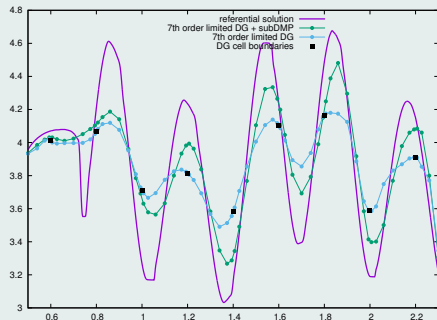## Shock acoustic-wave interaction problem

Figure : 7th order limited DG on 50 cells

Shock acoustic-wave interaction problem

(q) Zoom on $[-2, 0]$

(r) Zoom on $[0.5, 2.3]$

Figure : 7th order limited DG on 50 cells

Article in preparation

📄 F. VILAR, *A Posteriori Correction of High-Order Discontinuous Galerkin Scheme through Subcell Finite Volume Formulation and Flux Reconstruction.* JCP, 2018.

High-order Lagrangian hydrodynamics

📄 F. VILAR, C.-W. SHU AND P.-H. MAIRE, *Positivity-preserving cell-centered Lagrangian schemes for multi-material compressible flows: Form first-order to high-orders. Part II: The 2D case.* JCP, 2016.

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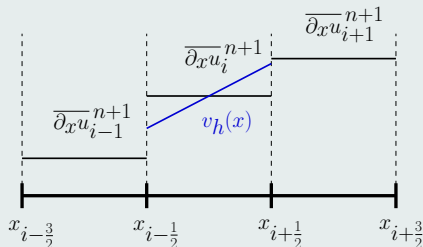
Relaxation of the DMP

- $v_L = \overline{\partial_x u_i}^{n+1} - \frac{\Delta x_i}{2} \overline{\partial_{xx} u_i}^{n+1}$

- $v_{\min \setminus \max} = \min \setminus \max (\overline{\partial_x u_i}^{n+1}, \overline{\partial_x u_{i-1}}^{n+1})$

- If $(v_L > \overline{\partial_x u_i}^{n+1})$ Then $\quad \alpha_L = \min(1, \dfrac{v_{\max} - \overline{\partial_x u_i}^{n+1}}{v_R - \overline{\partial_x u_i}^{n+1}})$

- If $(v_L < \overline{\partial_x u_i}^{n+1})$ Then $\quad \alpha_L = \min(1, \dfrac{v_{\min} - \overline{\partial_x u_i}^{n+1}}{v_R - \overline{\partial_x u_i}^{n+1}})$

- $v_R = \overline{\partial_x u_i}^{n+1} + \frac{\Delta x_i}{2} \overline{\partial_{xx} u_i}^{n+1}$

- $v_{\min \setminus \max} = \min \setminus \max (\overline{\partial_x u_i}^{n+1}, \overline{\partial_x u_{i+1}}^{n+1})$

- If $(v_R > \overline{\partial_x u_i}^{n+1})$ Then $\quad \alpha_R = \min(1, \dfrac{v_{\max} - \overline{\partial_x u_i}^{n+1}}{v_R - \overline{\partial_x u_i}^{n+1}})$

- If $(v_R < \overline{\partial_x u_i}^{n+1})$ Then $\quad \alpha_R = \min(1, \dfrac{v_{\min} - \overline{\partial_x u_i}^{n+1}}{v_R - \overline{\partial_x u_i}^{n+1}})$

Relaxation of the DMP

- $\alpha = \min(\alpha_L, \alpha_R)$
- If $(\alpha = 1)$ Then DMP is relaxed

Hierarchical limiter

- $v_h(x) = \overline{\partial_x u_i}^{n+1} + (x - x_i) \overline{\partial_{xx} u_i}^{n+1}$

📄 M. YANG and Z.J. WANG, *A parameter-free generalized moment limiter for high-order methods on unstructured grids.* AAMM., 2009.

📄 D. KUZMIN, *A vertex-based hierarchical slope limiter for p-adaptive discontinuous Galerkin methods.* J. of Comp. and Appl. Math., 2010.

Initial solution on $x \in [0, 1]$

- $u_0(x) = \sin(2\pi x)$
- Periodic boundary conditions
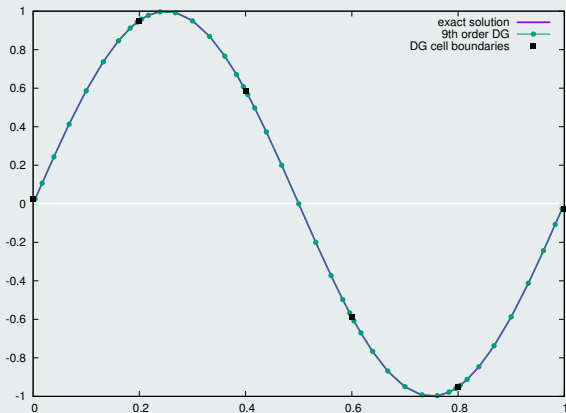
Figure : Linear advection with a 9th DG scheme and 5 cells after 1 period

Convergence rates

h	L_1		L_2	
	$E_{L_1}^h$	$q_{L_1}^h$	$E_{L_2}^h$	$q_{L_2}^h$
$\frac{1}{20}$	8.07E-11	9.00	8.97E-11	9.00
$\frac{1}{40}$	1.58E-13	9.00	1.75E-13	9.00
$\frac{1}{80}$	3.08E-16	-	3.42E-16	-

Table: Convergence rates for the linear advection case for a 9th order DG scheme
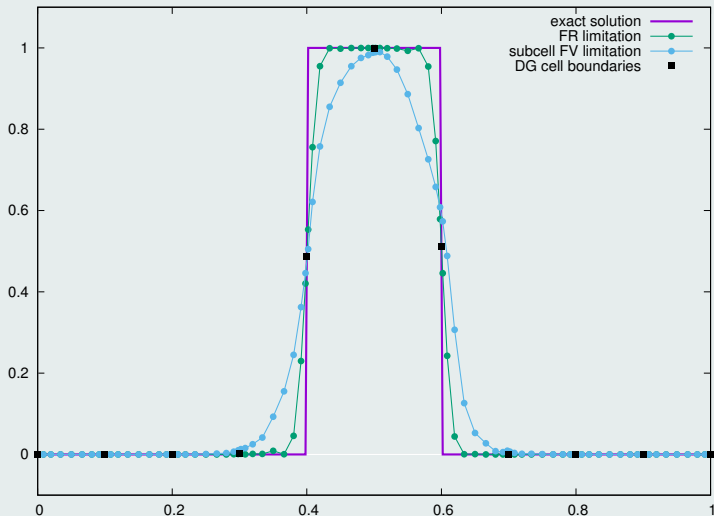
Linear advection of a square signal after 1 period

Figure : Comparison between flux reconstruction limitation and subcell finite volume limitation

Linear advection of a square signal after 10 periods
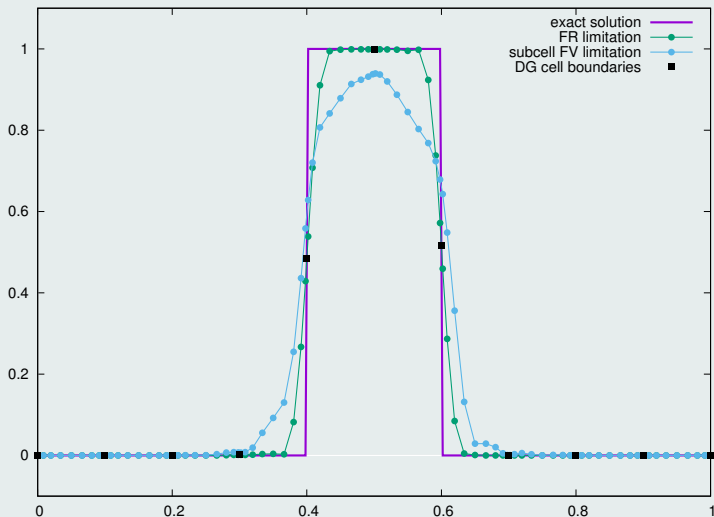
Figure : Comparison between flux reconstruction limitation and subcell finite volume limitation
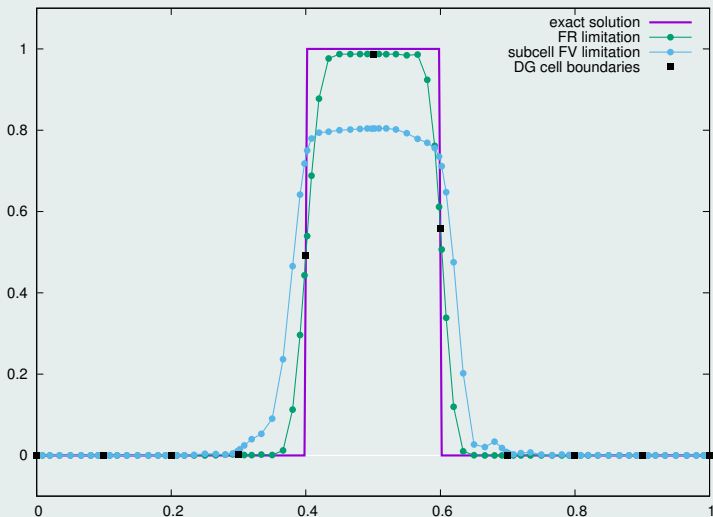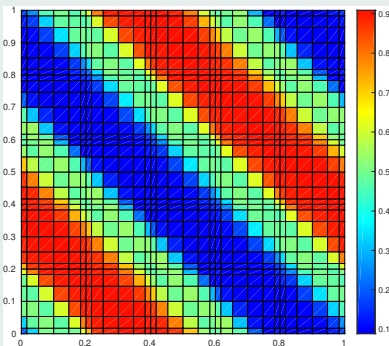
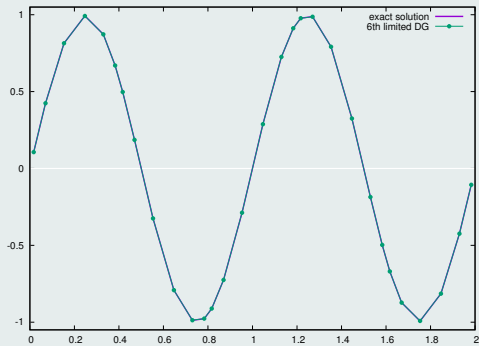# Linear advection of a square signal after 50 periods

Figure : Comparison between flux reconstruction limitation and subcell finite volume limitation

Initial solution on $(x, y) \in [0, 1]^2$

- $u_0(x, y) = \sin(2\pi(x + y))$
- Periodic boundary conditions

(s) Solution map

(t) Solution profile

Figure : Linear advection with a 6th DG scheme and 5x5 grid after 1 period

Convergence rates

	L_1		L_2	
h	$E_{L_1}^h$	$q_{L_1}^h$	$E_{L_2}^h$	$q_{L_2}^h$
$\frac{1}{5}$	2.10E-6	6.23	2.86E-6	6.24
$\frac{1}{10}$	2.79E-8	6.00	3.77E-8	6.00
$\frac{1}{20}$	3.36E-10	-	5.91E-10	-

Table: Convergence rates for the linear advection case for a 6th order DG scheme

Initial solution on $x \in [0, 1]$ for $\gamma = 3$

- $\rho_0(x) = 1 + 0.9999999\sin(\pi x), \quad u_0(x) = 0, \quad p_0(x) = (\rho_0(x))^\gamma$
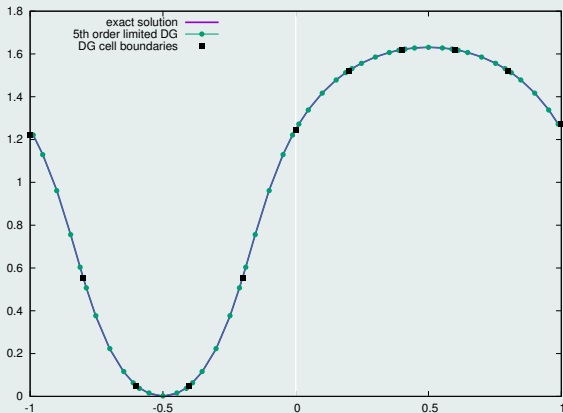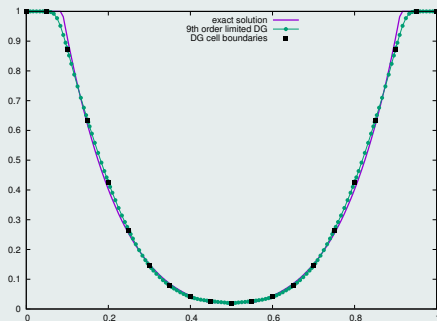- Periodic boundary conditions

Figure : Smooth flow problem with 5th DG scheme and 10 cells at $t = 0.1$
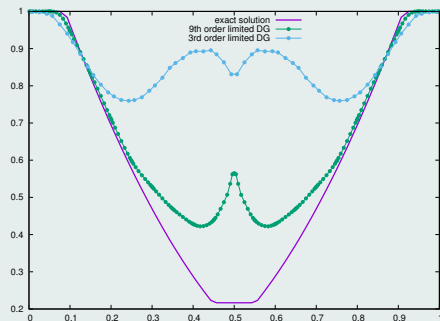
Convergence rates

	L_1		L_2	
h	$E_{L_1}^h$	$q_{L_1}^h$	$E_{L_2}^h$	$q_{L_2}^h$
$\frac{1}{20}$	1.48E-5	4.35	2.02E-5	4.18
$\frac{1}{40}$	9.09E-7	4.88	1.38E-6	4.87
$\frac{1}{80}$	3.09E-8	4.95	4.73E-8	4.86
$\frac{1}{160}$	1.00E-9	-	1.63E-9	-

Table: Convergence rates on the pressure for the Euler equation for a 5th order DG

Double rarefaction problem

(u) Density

(v) Internal energy

Figure : 9th order limited DG on 20 cells
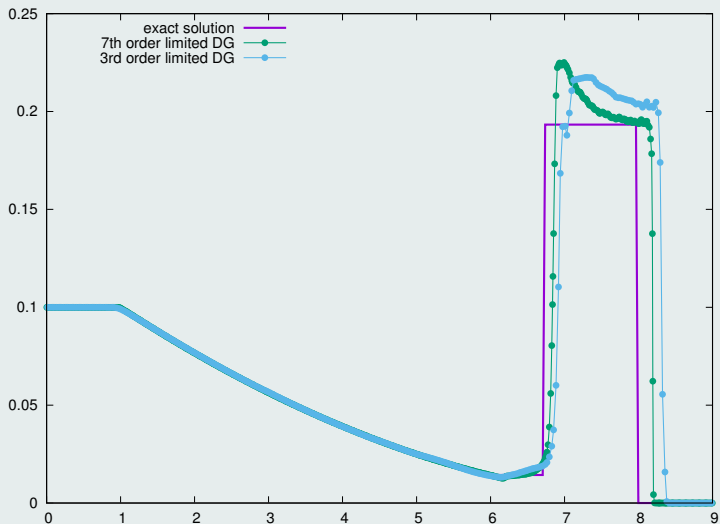
Leblanc shock tube problem

Figure : 3rd order vs 7th order limited DG on 100 cells
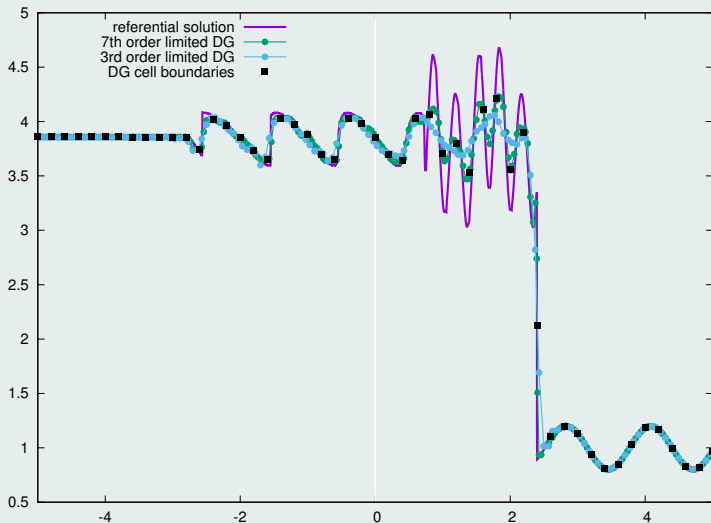
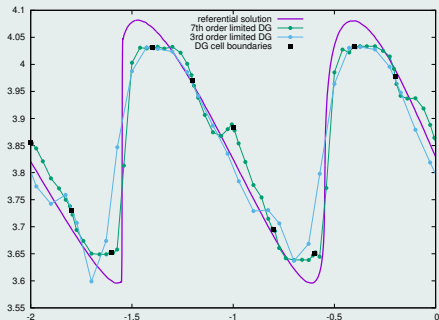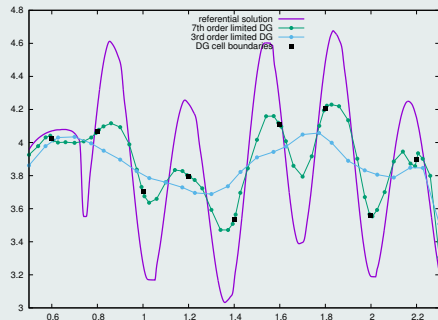## Shock acoustic-wave interaction problem

Figure : 3rd order vs 7th order limited DG on 50 cells

Shock acoustic-wave interaction problem

(w) Zoom on $[-2, 0]$

(x) Zoom on $[0.5, 2.3]$

Figure : 3rd order vs 7th order limited DG on 50 cells