DE LA RECHERCHE À L'INDUSTRIE

ECCOMAS 2012

High-order cell-centered DG scheme for Lagrangian hydrodynamics

CEA CESTA[1] and INRIA[2] | F. Vilar[1], P.-H. Maire[1], R. Abgrall[2]

10 SEPTEMBER 2012

www.cea.fr

Content

Content

Discontinuous Galerkin (DG)

- Natural extension of Finite Volume method
- Piecewise polynomial approximation of the solution in the cells
- High-order scheme to achieve high accuracy

- Local variational formulation
- Choice of the numerical fluxes (global L^2 stability, entropy inequality)
- Time discretization - TVD multistep Runge-Kutta

  C.-W. SHU, *Discontinuous Galerkin methods : General approach and stability*, 2008

- Limitation - vertex-based hierarchical slope limiters

  D. KUZMIN, *A vertex-based hierarchical slope limiter for p-adaptive discontinuous Galerkin methods* J. Comp. Appl. Math., 2009

  M. YANG AND Z.J. WANG, *A parameter-free generalized moment limiter for high-order methods on unstrucured grids* Adv. Appl. Math. Mech., 2009

Content

Cell-Centered Lagrangian schemes

Finite volume schemes on moving mesh

- J. K. Dukowicz : CAVEAT scheme
  A computer code for fluid dynamics problems with large distorsion and internal slip, 1986
- B. Després : GLACE scheme
  Lagrangian Gas Dynamics in Two Dimensions and Lagrangian systems, 2005
- P.-H. Maire : EUCCLHYD scheme
  A cell-centered Lagrangian scheme for two-dimensional compressible flow problems, 2007
- G. Kluth : Hyperelasticity
  Discretization of hyperelasticity with a cell-centered Lagrangian scheme, 2010
- S. Del Pino : Curvilinear Finite Volume method
  A curvilinear finite-volume method to solve compressible gas dynamics in semi-Lagrangian coordinates, 2010
- P. Hoch : Finite Volume method on unstructured conical meshes
  Extension of ALE methodoly to unstructured conical meshes, 2011

DG scheme on initial mesh

- R. Loubère : PhD thesis
  Une Méthode Particulaire Lagrangienne de type Galerkin Discontinu. Application à la mécanique des Fluides et l'Intéraction Laser/Plasma, 2002

Lagrangian and Eulerian descriptions

- Let's have a continuous mapping

$$\boldsymbol{x} = \boldsymbol{\Phi}(\boldsymbol{X}, t)$$

- $\boldsymbol{X}$ the Lagrangian (initial) coordinate
  $\boldsymbol{x}$ the Eulerian (actual) coordinate
  $\boldsymbol{N}$ the Lagrangian normal
  $\boldsymbol{n}$ the Eulerian normal
- $\mathrm{F} = \nabla_X \boldsymbol{\Phi} = \frac{\partial \boldsymbol{x}}{\partial X}$ the deformation gradient tensor, $J = \det \mathrm{F}$

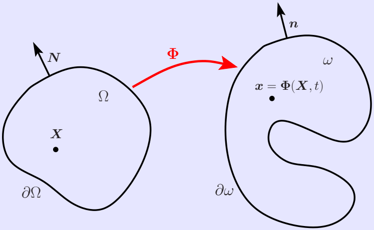


FIG.: Notation for the flow map.

- $\rho J = \rho^0$
- $\mathrm{d}v = J \mathrm{d}V$
- $\mathrm{d}\boldsymbol{x} = \mathrm{F}\mathrm{d}\boldsymbol{X}$
- $J\mathrm{F}^{-t}\boldsymbol{N}\mathrm{d}S = \boldsymbol{n}\mathrm{d}s$ Nanson formula

- $\boldsymbol{\nabla}_x P = \frac{1}{J}\nabla_X \cdot (P\, J\mathrm{F}^{-t})$
- $\nabla_x \cdot \boldsymbol{U} = \frac{1}{J}\nabla_X \cdot (J\mathrm{F}^{-1}\boldsymbol{U})$

Lagrangian and Eulerian descriptions

- $\mathrm{G} = J\mathrm{F}^{-\mathrm{t}}$ the cofactor matrix of F
- $\nabla_x \cdot G = \mathbf{0}$ Piola compatibility condition

$$\int_{\Omega_c} \nabla_x \cdot G \,\mathrm{d}V = \int_{\partial\Omega_c} \mathrm{G}\,\boldsymbol{N}\,\mathrm{d}S = \int_{\partial\omega_c} \boldsymbol{n}\,\mathrm{d}s = \mathbf{0}$$

- Gas dynamics system in Lagrangian formalism

$$\frac{\mathrm{d}\,\mathrm{F}}{\mathrm{d}t} = \nabla_X \boldsymbol{U}$$

$$\rho^0 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\right) - \nabla_X \cdot (\mathrm{G}^{\mathrm{t}}\boldsymbol{U}) = 0$$

$$\rho^0 \frac{\mathrm{d}\,\boldsymbol{U}}{\mathrm{d}t} + \nabla_X \cdot (P\,\mathrm{G}) = \mathbf{0}$$

$$\rho^0 \frac{\mathrm{d}\,E}{\mathrm{d}t} + \nabla_X \cdot (\mathrm{G}^{\mathrm{t}} P \boldsymbol{U}) = 0$$

- Thermodynamical closure EOS : $P = P(\rho, \varepsilon)$ where $\varepsilon = E - \frac{1}{2}\boldsymbol{U}^2$

Content

DG discretization general framework

DG discretization

- Let $\{\Omega_c\}_c$ be a partition of the domain Ω into polygonal cells
- $\{\sigma_k^c\}_{k=0\dots K}$ basis of $\mathbb{P}^\alpha(\Omega_c)$
- $\phi_h^c(\boldsymbol{X},t) = \sum_{k=0}^{K} \phi_k^c(t)\sigma_k^c(\boldsymbol{X})$ approximate function of $\phi(\boldsymbol{X},t)$ on Ω_c

Definitions

- Center of mass $\boldsymbol{\Xi}_c = (\Xi_c^X, \Xi_c^Y)^{\mathrm{t}} = \dfrac{1}{m_c}\displaystyle\int_{\Omega_c} \rho^0(\boldsymbol{X})\,\boldsymbol{X}\,\mathrm{d}V$,
  where m_c is the constant mass of the cell Ω_c
- The mean value $\langle\phi\rangle_c = \dfrac{1}{m_c}\displaystyle\int_{\Omega_c} \rho^0(\boldsymbol{X})\,\phi(\boldsymbol{X})\,\mathrm{d}V$
  of the function ϕ over the cell Ω_c
- The associated scalar product $\langle\phi,\psi\rangle_c = \displaystyle\int_{\Omega_c} \rho^0(\boldsymbol{X})\,\phi(\boldsymbol{X})\,\psi(\boldsymbol{X})\,\mathrm{d}V$

DG discretization general framework

Polynomial Taylor basis

- Taylor expansion on the cell, located at the center of mass Ξ_c
- $\sigma_0^c = 1$ and going further in space discretization, the $q+1$ basis functions of degree q, with $0 < q \leq \alpha$, write

$$\sigma^c_{\frac{q(q+1)}{2}+j} = \frac{1}{j!(q-j)!}\left[\left(\frac{X-\Xi_c^X}{\Delta X_c}\right)^{q-j}\left(\frac{Y-\Xi_c^Y}{\Delta Y_c}\right)^{j} - \left\langle\left(\frac{X-\Xi_c^X}{\Delta X_c}\right)^{q-j}\left(\frac{Y-\Xi_c^Y}{\Delta Y_c}\right)^{j}\right\rangle_c\right]$$

where $j = 0\ldots q$, $\Delta X_c = \frac{X_{max}-X_{min}}{2}$ and $\Delta Y_c = \frac{Y_{max}-Y_{min}}{2}$ with X_{max}, Y_{max}, X_{min}, Y_{min} the maximum and minimum coordinates in the cell Ω_c

Outcome

- Same basis functions whatever the shape of the cells
- $\left\langle \sigma_k^c \right\rangle_c = m_c\, \delta_{0k}$ where δ_{kl} is the Kronecker symbol
- $\left\langle \sigma_0^c, \sigma_k^c \right\rangle_c = 0,\ \forall k \neq 0$
- $\phi_0^c = \langle \phi \rangle_c$ the mass averaged value of the function ϕ over the cell Ω_c

Local variational formulations

- $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_c}\rho^0\left(\frac{1}{\rho}\right)\sigma_q^c\,\mathrm{d}V &= \sum_{k=0}^{K}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\right)_k^c\int_{\Omega_c}\rho^0\sigma_q^c\sigma_k^c\,\mathrm{d}V \\ &= -\int_{\Omega_c}\boldsymbol{U}\,.\,\mathrm{G}\,\nabla_X\sigma_q^c\,\mathrm{d}V + \int_{\partial\Omega_c}\overline{\boldsymbol{U}}\,.\,\sigma_q^c\,\mathrm{G}\boldsymbol{N}\mathrm{d}L\end{aligned}$$

- $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_c}\rho^0\boldsymbol{U}\sigma_q^c\,\mathrm{d}V &= \sum_{k=0}^{K}\frac{\mathrm{d}\,\boldsymbol{U}_k^c}{\mathrm{d}t}\int_{\Omega_c}\rho^0\sigma_q^c\sigma_k^c\,\mathrm{d}V \\ &= \int_{\Omega_c}P\,\mathrm{G}\,\nabla_X\sigma_q^c\,\mathrm{d}V - \int_{\partial\Omega_c}\overline{P}\,\sigma_q^c\,\mathrm{G}\boldsymbol{N}\mathrm{d}L\end{aligned}$$

- $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_c}\rho^0 E\sigma_q^c\,\mathrm{d}V &= \sum_{k=0}^{K}\frac{\mathrm{d}\,E_k^c}{\mathrm{d}t}\int_{\Omega_c}\rho^0\sigma_q^c\sigma_k^c\,\mathrm{d}V \\ &= \int_{\Omega_c}P\boldsymbol{U}\,.\,\mathrm{G}\,\nabla_X\sigma_q^c\,\mathrm{d}V - \int_{\partial\Omega_c}\overline{P\boldsymbol{U}}\,.\,\sigma_q^c\,\mathrm{G}\boldsymbol{N}\mathrm{d}L\end{aligned}$$

Local variational formulations

- $\int_{\Omega_c} \rho^0 \sigma_q^c \sigma_k^c \, \mathrm{d}V = \langle \sigma_q^c, \sigma_k^c \rangle_c$ generic coefficient of the symmetric positive definite mass matrix
- $\int_{\Omega_c} \rho^0 \, \sigma_k^c \, \mathrm{d}V = 0, \; \forall k \neq 0$ implies that the equations corresponding to mass averaged values are independent of the other polynomial basis components equations

- $\int_{\Omega_c} \boldsymbol{U} \, \boldsymbol{.} \, \mathsf{G} \, \nabla_X \sigma_q^c \, \mathrm{d}V$, $\int_{\Omega_c} P \, \mathsf{G} \, \nabla_X \sigma_q^c \, \mathrm{d}V$ and $\int_{\Omega_c} P\boldsymbol{U} \, \boldsymbol{.} \, \mathsf{G} \, \nabla_X \sigma_q^c \, \mathrm{d}V$ are evaluated through the use of a two-dimensional quadrature rule
- $\int_{\partial\Omega_c} \overline{\boldsymbol{U}} \, \boldsymbol{.} \, \sigma_q^c \, \mathsf{G} \boldsymbol{N} \mathrm{d}L$, $\int_{\partial\Omega_c} \overline{P} \, \sigma_q^c \, \mathsf{G} \boldsymbol{N} \mathrm{d}L$ and $\int_{\partial\Omega_c} \overline{P\boldsymbol{U}} \, \boldsymbol{.} \, \sigma_q^c \, \mathsf{G} \boldsymbol{N} \mathrm{d}L$ required a specific treatment to ensure the GCL

Entropic analysis

Entropic semi-discrete equation

- Fundamental assumption $\overline{P\,\boldsymbol{U}} = \overline{P}\,\overline{\boldsymbol{U}}$
- The use of variational formulations and Piola condition leads to

$$\int_{\Omega_c} \rho^0\,\theta \frac{\mathrm{d}\,\eta}{\mathrm{d}t}\,\mathrm{d}V = \int_{\partial\Omega_c} (\overline{P} - P)(\boldsymbol{U} - \overline{\boldsymbol{U}})\,.\,\mathsf{G}\boldsymbol{N}\mathrm{d}L,$$

where η is the specific entropy and θ the absolute temperature defined by means of the Gibbs identity

Entropic semi-discrete equation

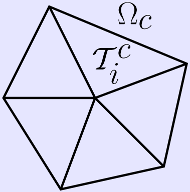
- A sufficient condition to satisfy $\int_{\Omega_c} \rho^0\,\theta \frac{\mathrm{d}\,\eta}{\mathrm{d}t}\,\mathrm{d}V \geq 0$ is

$$\overline{P} - P = -Z\,(\overline{\boldsymbol{U}} - \boldsymbol{U})\,.\,\frac{\mathsf{G}\boldsymbol{N}}{\|\mathsf{G}\boldsymbol{N}\|} = -Z\,(\overline{\boldsymbol{U}} - \boldsymbol{U})\,.\,\boldsymbol{n}, \qquad (2)$$

where $Z \geq 0$ has the physical dimension of a density times a velocity

Deformation gradient tensor discretization

- Requirements : Piola compatibility condition and geometry continuity
- Triangular decomposition $\Omega_c = \bigcup_{i=1}^{ntri} \mathcal{T}_i^c$
- F discretization by means of a mapping defined on triangular cells



- We develop $\mathbf{\Phi}$ on the Finite Element basis functions λ_p

$$\mathbf{\Phi}_h^i(\boldsymbol{X}, t) = \sum_p \lambda_p(\boldsymbol{X})\ \mathbf{\Phi}_p(t),$$

where the points p are control points including vertices in $\mathcal{T}_i$

- $\mathbf{\Phi}_p(t) = \mathbf{\Phi}(\boldsymbol{X}_p, t) = \boldsymbol{x}_p$
- $\mathrm{F} = \nabla_X \mathbf{\Phi} \Longrightarrow \mathrm{F}_i(\boldsymbol{X}, t) = \sum_p \mathbf{\Phi}_p(t) \otimes \boldsymbol{\nabla}_X \lambda_p(\boldsymbol{X})$
- $\dfrac{\mathrm{d}\,\mathbf{\Phi}_p}{\mathrm{d}t} = \boldsymbol{U}_p \Longrightarrow \dfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{F}_i(\boldsymbol{X}, t) = \sum_p \boldsymbol{U}_p(t) \otimes \boldsymbol{\nabla}_X \lambda_p(\boldsymbol{X})$

Content

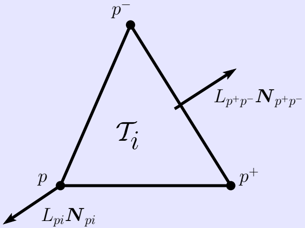
P_1 deformation gradient tensor discretization

- The chosen linear basis functions are the P_1 barycentric coordinate basis functions which write in a generic triangle $\mathcal{T}_i$

$$\lambda_p(\boldsymbol{X}) = \frac{1}{2|\mathcal{T}_i|}[X(Y_{p^+} - Y_{p^-}) - Y(X_{p^+} - X_{p^-}) + X_{p^+}Y_{p^-} - X_{p^-}Y_{p^+}],$$

where p, p^+ and p^- are the counterclockwise ordered triangle nodes and $|\mathcal{T}_i|$ the triangle volume

- $$\mathsf{F}_i(t) = \frac{1}{|\mathcal{T}_i|}\sum_{p\in\mathcal{P}(\mathcal{T}_i)} \boldsymbol{x}_p(t) \otimes L_{pi}\boldsymbol{N}_{pi},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{F}_i(t) = \frac{1}{|\mathcal{T}_i|}\sum_{p\in\mathcal{P}(\mathcal{T}_i)} \boldsymbol{U}_p(t) \otimes L_{pi}\boldsymbol{N}_{pi},$$
where $\mathcal{P}(\mathcal{T}_i)$ is the node set of $\mathcal{T}_i$



- $L_{pi}\boldsymbol{N}_{pi} = \frac{1}{2}(L_{p^-p}\boldsymbol{N}_{p^-p} + L_{pp^+}\boldsymbol{N}_{pp^+})$ the corner normal at node p in the initial configuration

Local boundary terms integration

Numerical fluxes linear approximation

- On face f_{pp^+}, $\overline{\psi}^{c}_{|_{pp^+}}(\zeta) = \psi^{+}_{pc}(1-\zeta) + \psi^{-}_{p^+c}\zeta$, where $\zeta \in [0,1]$ is the curvilinear abscissa
- Hence

$$\overline{\boldsymbol{U}}_{|_{pp^+}}(\zeta) = \boldsymbol{U}_p(1-\zeta) + \boldsymbol{U}_{p^+}\zeta$$

$$\overline{P}^{c}_{|_{pp^+}}(\zeta) = P^{+}_{pc}(1-\zeta) + P^{-}_{p^+c}\zeta$$

$$\overline{P\boldsymbol{U}}^{c}_{|_{pp^+}}(\zeta) = (P\boldsymbol{U})^{+}_{pc}(1-\zeta) + (P\boldsymbol{U})^{-}_{p^+c}\zeta$$

- The basis function σ^c_q being linear over Ω_c

$$\sigma^{c}_{q|_{pp^+}}(\zeta) = \sigma^c_q(\boldsymbol{X}_p)(1-\zeta) + \sigma^c_q(\boldsymbol{X}_{p^+})\zeta$$

Fundamental assumption

- $$\overline{P\,\boldsymbol{U}} = \overline{P}\,\overline{\boldsymbol{U}} \Longrightarrow (P\boldsymbol{U})^{\pm}_{pc} = P^{\pm}_{pc}\,\boldsymbol{U}_p$$

Local boundary terms integration

Analytical integration

- $$\int_{\partial\Omega_c} \overline{\psi}\sigma_q^c \, \mathsf{G}\boldsymbol{N} \mathrm{d}L = \sum_{p\in\mathcal{P}(c)} \left(\int_0^1 \overline{\psi}_{|_{pp^+}}(\zeta)\, \sigma^c_{q|_{pp^+}}(\zeta)\, \mathrm{d}\zeta \right) \mathsf{G}_{|_{pp^+}} L_{pp^+} \boldsymbol{N}_{pp^+},$$
  where $\mathsf{G}_{|_{pp^+}}$ is the constant value of tensor G on face f_{pp^+}
- $\mathsf{G}_{|_{pp^+}} L_{pp^+} \boldsymbol{N}_{pp^+} = l_{pp^+} \boldsymbol{n}_{pp^+}$ Eulerian normal of face f_{pp^+}
- $$\begin{aligned}\int_{\partial\Omega_c} \overline{\psi}\sigma_q^c \, \mathsf{G}\boldsymbol{N} \mathrm{d}L &= \sum_{p\in\mathcal{P}(c)} \left(\int_0^1 \overline{\psi}\sigma^c_{q|_{pp^+}}(\zeta)\, \mathrm{d}\zeta \right) l_{pp^+} \boldsymbol{n}_{pp^+} \\ &= \sum_{p\in\mathcal{P}(c)} \frac{1}{6}[\psi^+_{pc}(2\sigma^c_q(\boldsymbol{X}_p) + \sigma^c_q(\boldsymbol{X}_{p^+})) l_{pp^+} \boldsymbol{n}_{pp^+} \\ &\quad + \psi^-_{p^+c}(2\sigma^c_q(\boldsymbol{X}_{p^+}) + \sigma^c_q(\boldsymbol{X}_p)) l_{pp^+} \boldsymbol{n}_{pp^+}] \\ &= \sum_{p\in\mathcal{P}(c)} \frac{1}{6}[\psi^+_{pc}(2\sigma^c_q(\boldsymbol{X}_p) + \sigma^c_q(\boldsymbol{X}_{p^+})) l_{pp^+} \boldsymbol{n}_{pp^+} \\ &\quad + \psi^-_{pc}(2\sigma^c_q(\boldsymbol{X}_p) + \sigma^c_q(\boldsymbol{X}_{p^-})) l_{p^-p} \boldsymbol{n}_{p^-p}]\end{aligned}$$

Semi-discrete equations

- Half-Left and half-right corner normals $l_{pc}^{\pm}\boldsymbol{n}_{pc}^{\pm}$
$$l_{pc}^{-}\boldsymbol{n}_{pc}^{-} = \frac{1}{2} l_{p^-p}\boldsymbol{n}_{p^-p} \quad \text{and} \quad l_{pc}^{+}\boldsymbol{n}_{pc}^{+} = \frac{1}{2} l_{pp^+}\boldsymbol{n}_{pp^+}$$
- The weighted corner normals
$$l_{pc}^{q}\boldsymbol{n}_{pc}^{q} = l_{pc}^{-,q}\boldsymbol{n}_{pc}^{-} + l_{pc}^{+,q}\boldsymbol{n}_{pc}^{+} \quad \text{where} \quad l_{pc}^{\pm,q} = \frac{1}{3}\left(2\sigma_q^c(\boldsymbol{X}_p) + \sigma_q^c(\boldsymbol{X}_{p^\pm})\right)\, l_{pc}^{\pm}$$
- The q^{th} moment of the subcell forces
$$\boldsymbol{F}_{pc}^{q} = P_{pc}^{-}\, l_{pc}^{-,q}\boldsymbol{n}_{pc}^{-} + P_{pc}^{+}\, l_{pc}^{+,q}\boldsymbol{n}_{pc}^{+}$$

- The semi-discrete equations on the specific volume, momentum and total energy successive moments, in respect with the GCL, write
$$\int_{\Omega_c} \rho^0 \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{\rho})\sigma_q^c \,\mathrm{d}V = -\sum_{i=1}^{ntri} \mathrm{G}_i^c \nabla_X \sigma_q^c \,.\, \int_{\mathcal{T}_i^c} \boldsymbol{U}\mathrm{d}V + \sum_{p\in\mathcal{P}(c)} \boldsymbol{U}_p \,.\, l_{pc}^{q}\boldsymbol{n}_{pc}^{q}$$
$$\int_{\Omega_c} \rho^0 \frac{\mathrm{d}\,\boldsymbol{U}}{\mathrm{d}t}\sigma_q^c \,\mathrm{d}V = \sum_{i=1}^{ntri} \mathrm{G}_i^c \nabla_X \sigma_q^c \int_{\mathcal{T}_i^c} P\mathrm{d}V - \sum_{p\in\mathcal{P}(c)} \boldsymbol{F}_{pc}^{q}$$
$$\int_{\Omega_c} \rho^0 \frac{\mathrm{d}\,E}{\mathrm{d}t}\sigma_q^c \,\mathrm{d}V = \sum_{i=1}^{ntri} \mathrm{G}_i^c \nabla_X \sigma_q^c \,.\, \int_{\mathcal{T}_i^c} P\,\boldsymbol{U}\mathrm{d}V - \sum_{p\in\mathcal{P}(c)} \boldsymbol{U}_p \,.\, \boldsymbol{F}_{pc}^{q}$$

Nodal solvers

- Fundamental identity on the cell Ω_c

$$\overline{P} = P_h^c - Z_c\,(\overline{\boldsymbol{U}} - \boldsymbol{U}_h^c)\,.\,\boldsymbol{n},$$

where $Z_c = \rho_c\, a_c$ is the acoustic impedance

- Using this expression to calculate $\boldsymbol{F}_{pc}^q$ leads to

$$\begin{aligned}\boldsymbol{F}_{pc}^q &= P_{pc}^-\, l_{pc}^{-,q}\boldsymbol{n}_{pc}^- + P_{pc}^+\, l_{pc}^{+,q}\boldsymbol{n}_{pc}^+ \\ &= P_h^c(\boldsymbol{X}_p, t)\, l_{pc}^q\boldsymbol{n}_{pc}^q \quad - Z_c\left[(\boldsymbol{U}_p - \boldsymbol{U}_h^c(\boldsymbol{X}_p, t))\,.\,\boldsymbol{n}_{pc}^-\right] l_{pc}^{-,q}\boldsymbol{n}_{pc}^- \\ &\qquad\qquad - Z_c\left[(\boldsymbol{U}_p - \boldsymbol{U}_h^c(\boldsymbol{X}_p, t))\,.\,\boldsymbol{n}_{pc}^+\right] l_{pc}^{+,q}\boldsymbol{n}_{pc}^+\end{aligned}$$

- Finally, the q^{th} moment of the subcell force writes

$$\boldsymbol{F}_{pc}^q = P_h^c(\boldsymbol{X}_p, t)\, l_{pc}^q\boldsymbol{n}_{pc}^q - \mathrm{M}_{pc}^q\,(\boldsymbol{U}_p - \boldsymbol{U}_h^c(\boldsymbol{X}_p, t)),$$

where $\mathrm{M}_{pc}^q = Z_c\,\left(l_{pc}^{-,q}\boldsymbol{n}_{pc}^- \otimes \boldsymbol{n}_{pc}^- + l_{pc}^{+,q}\boldsymbol{n}_{pc}^+ \otimes \boldsymbol{n}_{pc}^+\right)$

Nodal solvers

- To be conservative in total energy and momentum over the whole domain, $\sum_{c\in\mathcal{C}(p)} \boldsymbol{F}_{pc} = \boldsymbol{0}$ and thus

$$(\sum_{c\in\mathcal{C}(p)} \mathrm{M}_{pc})\, \boldsymbol{U}_p = \sum_{c\in\mathcal{C}(p)} \left[P_h^c(\boldsymbol{X}_p, t)\, l_{pc}\boldsymbol{n}_{pc} + \mathrm{M}_{pc}\, \boldsymbol{U}_h^c(\boldsymbol{X}_p, t)\right],$$

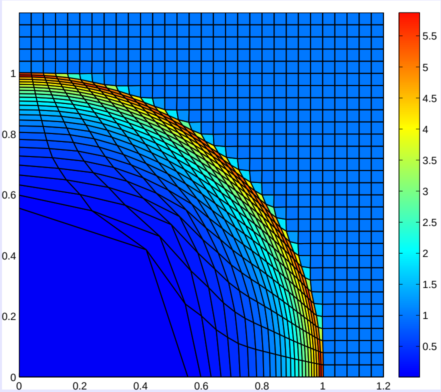
where $\mathrm{M}_{pc} = Z_c\,(l_{pc}^{+}\boldsymbol{n}_{pc}^{+} \otimes \boldsymbol{n}_{pc}^{+} + l_{pc}^{-}\boldsymbol{n}_{pc}^{-} \otimes \boldsymbol{n}_{pc}^{-})$ are positive semi-definite matrices with a physical dimension of a density times a velocity

First moment equations

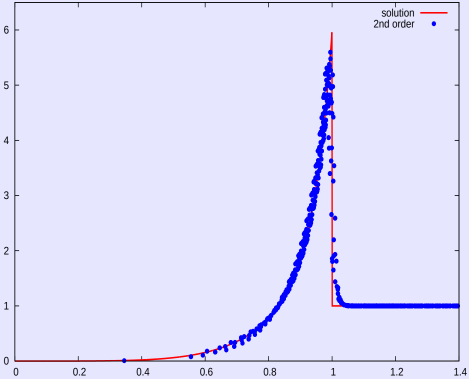
- $m_c \dfrac{\mathrm{d}}{\mathrm{d}t}(\dfrac{1}{\rho})_0^c = \sum_{p\in\mathcal{P}(c)} \boldsymbol{U}_p \,.\, l_{pc}\boldsymbol{n}_{pc}$
- $m_c \dfrac{\mathrm{d}\,\boldsymbol{U}_0^c}{\mathrm{d}t} = -\sum_{p\in\mathcal{P}(c)} \boldsymbol{F}_{pc}$
- $m_c \dfrac{\mathrm{d}\,E_0^c}{\mathrm{d}t} = -\sum_{p\in\mathcal{P}(c)} \boldsymbol{U}_p \,.\, \boldsymbol{F}_{pc}$

We recover the EUCCLHYD scheme

Sedov point blast problem on a Cartesian grid



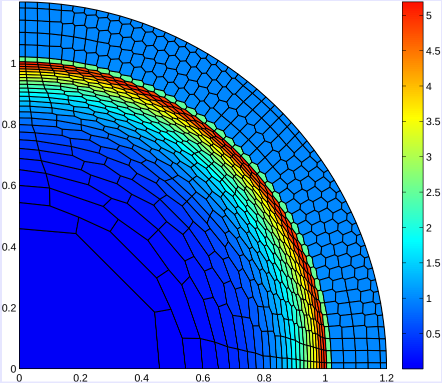
(a) Second-order scheme with limitation.



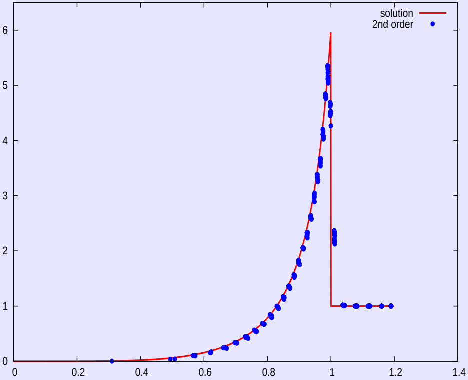
(b) Density profiles comparison.

FIG.: Point blast Sedov problem on a Cartesian grid made of 30 $\times$ 30 cells : density.

Sedov point blast problem on a polygonal grid



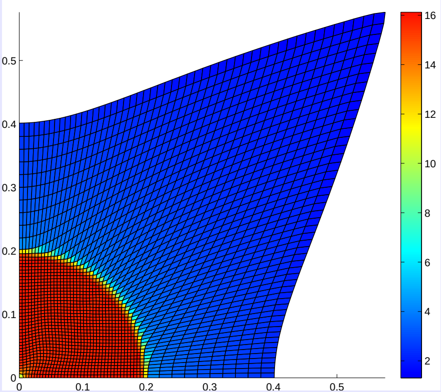
(a) Second-order scheme with limitation.



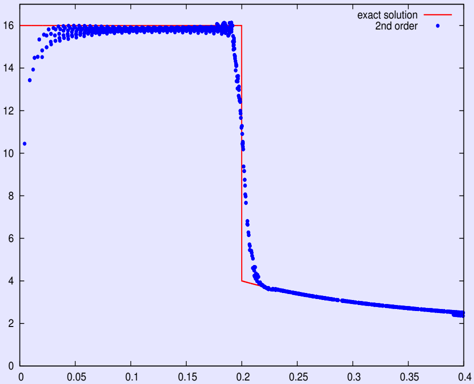
(b) Density profiles comparison.

FIG.: Point blast Sedov problem on a unstructured grid made of 775 polygonal cells : density map.

Noh problem



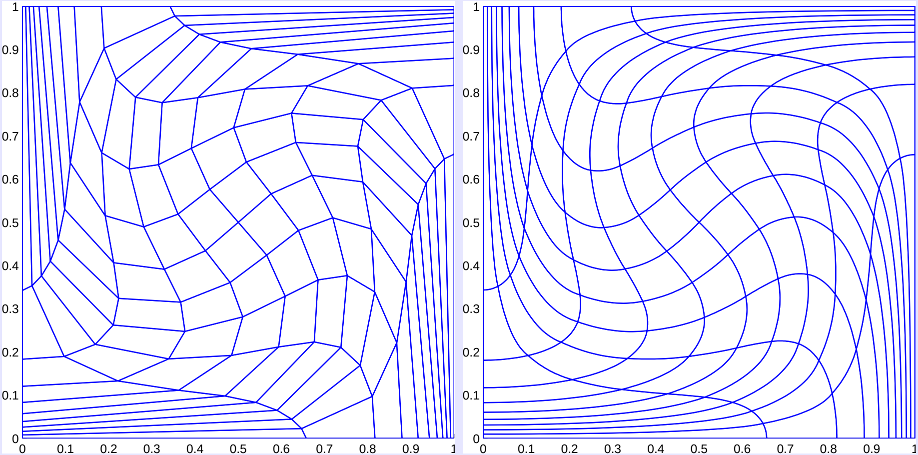
(a) Second-order scheme with limitation.



(b) Density profiles comparison.

FIG.: Noh problem on a Cartesian grid made of 50 $\times$ 50 cells : density.

Taylor-Green vortex problem



(a) Second-order scheme.

(b) Exact solution.

FIG.: Motion of a 10 × 10 Cartesian mesh through a T.-G. vortex, at $t = 0.75$.

Rate of convergence computed on the pressure in the case of the Taylor-Green vortex

	L_1		L_2		L_∞	
h	$E^h_{L_1}$	$q^h_{L_1}$	$E^h_{L_2}$	$q^h_{L_2}$	$E^h_{L_\infty}$	$q^h_{L_\infty}$
$\frac{1}{20}$	8.98E-3	1.88	1.51E-2	1.75	6.73E-2	1.27
$\frac{1}{40}$	2.44E-3	1.94	4.48E-3	1.95	2.79E-2	1.68
$\frac{1}{80}$	6.36E-4	2.00	1.16E-3	2.00	8.68E-3	1.95
$\frac{1}{160}$	1.59E-4	2.01	2.90E-4	2.01	2.24E-3	2.01
$\frac{1}{320}$	3.94E-5	-	7.18E-5	-	5.54E-4	-

TAB.: Second-order DG scheme without limitation at time $t = 0.6$.

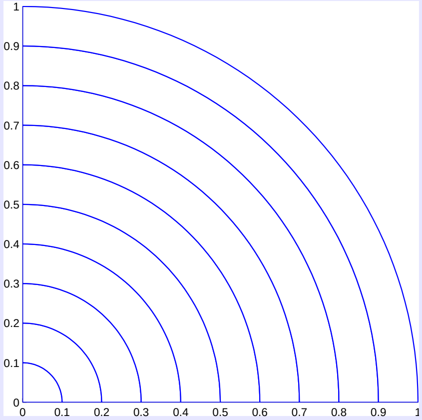
	L_1		L_2		L_∞	
h	$E^h_{L_1}$	$q^h_{L_1}$	$E^h_{L_2}$	$q^h_{L_2}$	$E^h_{L_\infty}$	$q^h_{L_\infty}$
$\frac{1}{20}$	1.99E-2	2.33	2.92E-2	2.03	8.27E-2	1.34
$\frac{1}{40}$	3.96E-3	2.25	7.16E-3	2.20	3.26E-2	1.61
$\frac{1}{80}$	8.31E-4	2.17	1.56E-3	2.15	1.07E-2	1.52
$\frac{1}{160}$	1.85E-4	2.11	3.52E-4	2.14	3.73E-3	2.41
$\frac{1}{320}$	4.28E-5	-	8.01E-5	-	7.01E-4	-

TAB.: Second-order DG scheme with limitation at time $t = 0.6$.

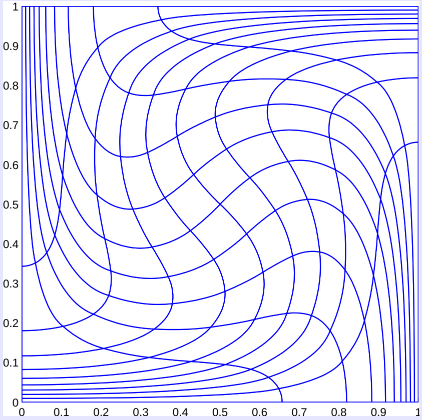
Content

Curvilinear elements motivation

Circular polar grid : 10 $\times$ 1 cells



Taylor-Green exact motion



V. DOBREV, T. ELLIS, T. KOLEV AND R. RIEBEN, *High Order Curvilinear Finite Elements for Lagrangian Hydrodynamics. Part I : General Framework*, 2010. Presentation available at `https://computation.llnl.gov/casc/blast/blast.html`

Geometric consideration

- The P_2 quadratic mapping function writes

$$\boldsymbol{x} = \boldsymbol{\Phi}(\boldsymbol{X}, t) = \sum_p \boldsymbol{x}_p(t)\, \mu_p(\boldsymbol{X}),$$

where the points p are the triangular nodes and the control points Q of the Bezier edges, and the P_2 barycentric coordinate functions μ_p write

$$\mu_p = (\lambda_p)^2,\ \mu_{p^+} = (\lambda_{p^+})^2,\ \mu_{p^-} = (\lambda_{p^-})^2,$$
$$\mu_Q = 2\lambda_p\lambda_{p^+},\ \mu_{Q^+} = 2\lambda_{p^+}\lambda_{p^-},\ \mu_{Q^-} = 2\lambda_{p^-}\lambda_p,$$

where the functions λ_l, with $l \in \{p, p^+, p^-\}$, are the P_1 Finite Elements linear basis functions

- Finally, the quadratic mapping expresses as

$$\boldsymbol{\Phi}(\boldsymbol{X}, t) = \sum_{p \in \mathcal{P}(\mathcal{T}_i)} \left[\boldsymbol{x}_p(t)\, (\lambda_p(\boldsymbol{X}))^2 + 2\boldsymbol{x}_Q(t)\, \lambda_p(\boldsymbol{X})\lambda_p^+(\boldsymbol{X})\right]$$

Geometric consideration

$$\mathrm{F}_i(\boldsymbol{X},t) = \frac{2}{|\mathcal{T}_c|} \sum_{p\in\mathcal{P}(\mathcal{T}_i)} \lambda_p \left[\boldsymbol{x}_p \otimes L_{pc}\boldsymbol{N}_{pc} + \boldsymbol{x}_Q \otimes L_{p^+c}\boldsymbol{N}_{p^+c} + \boldsymbol{x}_Q^- \otimes L_{p^-c}\boldsymbol{N}_{p^-c}\right]$$

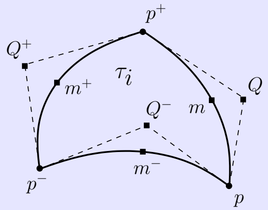
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{F}_i(\boldsymbol{X},t) = \frac{2}{|\mathcal{T}_i|} \sum_{p\in\mathcal{P}(\mathcal{T}_i)} \lambda_p \left[\boldsymbol{U}_p \otimes L_{pc}\boldsymbol{N}_{pc} + \boldsymbol{U}_Q \otimes L_{p^+c}\boldsymbol{N}_{p^+c} + \boldsymbol{U}_Q^- \otimes L_{p^-c}\boldsymbol{N}_{p^-c}\right]$$

where $\boldsymbol{U}_Q = \dfrac{4\boldsymbol{U}_m - \boldsymbol{U}_p - \boldsymbol{U}_{p^+}}{2}$ and $L_{pc}\boldsymbol{N}_{pc} = \frac{1}{2}(\boldsymbol{X}_{p^+} - \boldsymbol{X}_{p^-}) \times \boldsymbol{e}_Z$

- Given p, Q and p^+, and ζ in $[0,1]$, we define the Bezier curve as

$$\boldsymbol{x}(\zeta) = (1-\zeta)^2\boldsymbol{x}_p + 2\zeta(1-\zeta)\boldsymbol{x}_Q + \zeta^2\boldsymbol{x}_{p^+}$$

- Midpoint $\boldsymbol{x}_m = \boldsymbol{x}(\frac{1}{2}) = \dfrac{2\boldsymbol{x}_Q + \boldsymbol{x}_p + \boldsymbol{x}_{p^+}}{4}$



- $\boldsymbol{t}\mathrm{d}l = \dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\zeta}\,\mathrm{d}\zeta = 2\left((1-\zeta)(\boldsymbol{x}_Q - \boldsymbol{x}_p) + \zeta(\boldsymbol{x}_{p^+} - \boldsymbol{x}_Q)\right)\mathrm{d}\zeta$

Local boundary terms integration

Numerical fluxes quadratic approximation

- On face f_{pp^+}

$$\boldsymbol{U}_{|_{pp^+}}(\zeta) = (1-\zeta)(1-2\zeta)\boldsymbol{U}_p + 4\zeta(1-\zeta)\boldsymbol{U}_m + \zeta(2\zeta-1)\boldsymbol{U}_{p^+}$$

$$\overline{P}^{\,c}_{|_{pp^+}}(\zeta) = (1-\zeta)(1-2\zeta)P^+_{pc} + 4\zeta(1-\zeta)P_{mc} + \zeta(2\zeta-1)P^-_{p^+c}$$

$$\overline{P\boldsymbol{U}}^{\,c}_{|_{pp^+}}(\zeta) = (1-\zeta)(1-2\zeta)(P\boldsymbol{U})^+_{pc} + 4\zeta(1-\zeta)(P\boldsymbol{U})_{mc} + \zeta(2\zeta-1)(P\boldsymbol{U})^-_{p^+c}$$

- The basis function σ^c_q being quadratic over Ω_c

$$\sigma^c_{q|_{pp^+}}(\zeta) = (1-\zeta)(1-2\zeta)\,\sigma^c_q(\boldsymbol{X}_p) + 4\zeta(1-\zeta)\,\sigma^c_q(\boldsymbol{X}_m) + \zeta(2\zeta-1)\,\sigma^c_q(\boldsymbol{X}_{p^+})$$

Fundamental assumption

- $\overline{P\,\boldsymbol{U}} = \overline{P}\,\overline{\boldsymbol{U}} \Longrightarrow (P\boldsymbol{U})^{\pm}_{pc} = P^{\pm}_{pc}\,\boldsymbol{U}_p$ and $(P\boldsymbol{U})_{mc} = P_{mc}\,\boldsymbol{U}_m$

Normal and subcell forces definitions

$$l_{pc}^{q}\boldsymbol{n}_{pc}^{q} = l_{pc}^{-,q}\boldsymbol{n}_{pc}^{-,q} + l_{pc}^{+,q}\boldsymbol{n}_{pc}^{+,q}$$

$$l_{pc}^{-,q}\boldsymbol{n}_{pc}^{-,q} = \frac{1}{10}\left[(6\,\sigma_q^c(\boldsymbol{X}_p) + 4\,\sigma_q^c(\boldsymbol{X}_{m^-}))l_{Q^-p}\boldsymbol{n}_{Q^-p} + (\sigma_q^c(\boldsymbol{X}_p) - \sigma_q^c(\boldsymbol{X}_{p^-}))l_{p^-p}\boldsymbol{n}_{p^-p}\right]$$

$$l_{pc}^{+,q}\boldsymbol{n}_{pc}^{+,q} = \frac{1}{10}\left[(6\,\sigma_q^c(\boldsymbol{X}_p) + 4\,\sigma_q^c(\boldsymbol{X}_m))l_{pQ}\boldsymbol{n}_{pQ} + (\sigma_q^c(\boldsymbol{X}_p) - \sigma_q^c(\boldsymbol{X}_{p^+}))l_{pp^+}\boldsymbol{n}_{pp^+}\right]$$

$$\boldsymbol{F}_{pc}^{q} = P_{pc}^{-}\,l_{pc}^{-,q}\boldsymbol{n}_{pc}^{-,q} + P_{pc}^{+}\,l_{pc}^{+,q}\boldsymbol{n}_{pc}^{+,q}$$

$$l_{mc}^{q}\boldsymbol{n}_{mc}^{q} = l_{mc}^{-,q}\boldsymbol{n}_{mc}^{-} + l_{mc}^{+,q}\boldsymbol{n}_{mc}^{+}$$

$$l_{mc}^{-,q}\boldsymbol{n}_{mc}^{-} = \frac{1}{5}\left(4\,\sigma_q^c(\boldsymbol{X}_m) + \sigma_q^c(\boldsymbol{X}_p)\right)\,l_{pQ}\boldsymbol{n}_{pQ}$$

$$l_{mc}^{+,q}\boldsymbol{n}_{mc}^{+} = \frac{1}{5}\left(4\,\sigma_q^c(\boldsymbol{X}_m) + \sigma_q^c(\boldsymbol{X}_{p^+})\right)\,l_{Qp^+}\boldsymbol{n}_{Qp^+}$$

$$\boldsymbol{F}_{mc}^{q} = P_{mc}\,l_{mc}^{q}\boldsymbol{n}_{pc}^{q}$$

Case of $q = 0$

$$l_{pc}\boldsymbol{n}_{pc} = l_{Q^-Q}\boldsymbol{n}_{Q^-Q}$$

$$l_{mc}\boldsymbol{n}_{mc} = l_{pp^+}\boldsymbol{n}_{pp^+}$$

- $\displaystyle\int_{\partial\Omega_c} \overline{\boldsymbol{U}}\,.\,\mathrm{G}\boldsymbol{N}\mathrm{d}L = \sum_{p\in\mathcal{P}(c)} \frac{1}{3}\boldsymbol{U}_p\,.\,l_{Q^-Q}\boldsymbol{n}_{Q^-Q} + \sum_{m\in\mathcal{M}(c)} \frac{2}{3}\boldsymbol{U}_m\,.\,l_{pp^+}\boldsymbol{n}_{pp^+}$

Normal and subcell forces definitions

- The semi-discrete equations on the specific volume, momentum and total energy successive moments, ensuring the GCL, write

$$\int_{\Omega_c} \rho^0 \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{\rho})\sigma_q^c \,\mathrm{d}V = -\sum_{i=1}^{ntri} \int_{\mathcal{T}_i^c} \boldsymbol{U} \,.\, \mathrm{G}\nabla_X \sigma_q^c \mathrm{d}V + \sum_{p\in\mathcal{P}(c)} \frac{1}{3} \boldsymbol{U}_p \,.\, l_{pc}^q \boldsymbol{n}_{pc}^q + \sum_{m\in\mathcal{M}(c)} \frac{2}{3} \boldsymbol{U}_m \,.\, l_{mc}^q \boldsymbol{n}_{mc}^q$$

$$\int_{\Omega_c} \rho^0 \frac{\mathrm{d}\,\boldsymbol{U}}{\mathrm{d}t} \sigma_q^c \,\mathrm{d}V = \sum_{i=1}^{ntri} \int_{\mathcal{T}_i^c} P\mathrm{G}\nabla_X \sigma_q^c \mathrm{d}V - \sum_{p\in\mathcal{P}(c)} \frac{1}{3} \boldsymbol{F}_{pc}^q - \sum_{m\in\mathcal{M}(c)} \frac{2}{3} \boldsymbol{F}_{mc}^q$$

$$\int_{\Omega_c} \rho^0 \frac{\mathrm{d}\,E}{\mathrm{d}t} \sigma_q^c \,\mathrm{d}V = \sum_{i=1}^{ntri} \int_{\mathcal{T}_i^c} P\boldsymbol{U} \,.\, \mathrm{G}\nabla_X \sigma_q^c \mathrm{d}V - \sum_{p\in\mathcal{P}(c)} \frac{1}{3} \boldsymbol{U}_p \,.\, \boldsymbol{F}_{pc}^q - \sum_{m\in\mathcal{M}(c)} \frac{2}{3} \boldsymbol{U}_m \,.\, \boldsymbol{F}_{mc}^q$$

- $\overline{P} = P_h^c - Z_c\,(\overline{\boldsymbol{U}} - \boldsymbol{U}_h^c) \,.\, \boldsymbol{n}$
- $\boldsymbol{F}_{pc}^q = P_h^c(\boldsymbol{X}_p, t)\, l_{pc}^q \boldsymbol{n}_{pc}^q - \mathrm{M}_{pc}^q\,(\boldsymbol{U}_p - \boldsymbol{U}_h^c(\boldsymbol{X}_p, t))$,

  where $\mathrm{M}_{pc}^q = Z_c\,\left(l_{pc}^{-,q} \boldsymbol{n}_{pc}^{-,q} \otimes \boldsymbol{n}_{pc}^{-} + l_{pc}^{+,q} \boldsymbol{n}_{pc}^{+,q} \otimes \boldsymbol{n}_{pc}^{+} \right)$
- $\boldsymbol{F}_{mc}^q = P_h^c(\boldsymbol{X}_m, t)\, l_{mc}^q \boldsymbol{n}_{mc}^q - \mathrm{M}_{mc}^q\,(\boldsymbol{U}_m - \boldsymbol{U}_h^c(\boldsymbol{X}_m, t))$,

  where $\mathrm{M}_{mc}^q = Z_c\, l_{mc}^q \boldsymbol{n}_{mc}^q \otimes \boldsymbol{n}_{mc}$

Nodal and midpoint solvers

- Fundamental identity on the cell Ω_c

$$\overline{P} = P_h^c - Z_c\,(\overline{\boldsymbol{U}} - \boldsymbol{U}_h^c)\,.\,\boldsymbol{n},$$

where $Z_c = \rho_c\, a_c$ is the acoustic impedance

- Using this expression to calculate $\boldsymbol{F}_{pc}^q$ leads to

$$\boldsymbol{F}_{pc}^q = P_h^c(\boldsymbol{X}_p, t)\, l_{pc}^q \boldsymbol{n}_{pc}^q - \mathrm{M}_{pc}^q\,(\boldsymbol{U}_p - \boldsymbol{U}_h^c(\boldsymbol{X}_p, t)),$$

where $\mathrm{M}_{pc}^q = Z_c\,\left(l_{pc}^{-,q} \boldsymbol{n}_{pc}^{-,q} \otimes \boldsymbol{n}_{pc}^{-} + l_{pc}^{+,q} \boldsymbol{n}_{pc}^{+,q} \otimes \boldsymbol{n}_{pc}^{+} \right)$

- Regarding the midpoint subcell forces, $\boldsymbol{F}_{mc}^q$ writes

$$\boldsymbol{F}_{mc}^q = P_h^c(\boldsymbol{X}_m, t)\, l_{mc}^q \boldsymbol{n}_{mc}^q - \mathrm{M}_{mc}^q\,(\boldsymbol{U}_m - \boldsymbol{U}_h^c(\boldsymbol{X}_m, t)),$$

where the M_{mc}^q matrices are defined as

$$\mathrm{M}_{mc}^q = Z_c\, l_{mc}^q \boldsymbol{n}_{mc}^q \otimes \boldsymbol{n}_{mc}$$

Nodal solver

- To be conservative in total energy and momentum over the whole domain, we set the following sufficient conditions

$$\sum_{c\in\mathcal{C}(p)} \boldsymbol{F}_{pc} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{F}_{mL} + \boldsymbol{F}_{mR} = \boldsymbol{0},$$

  where $\mathcal{C}(p)$ is the set of cells surrounding the p node, and Ω_L and Ω_R the two cells surrounding the midpoint m

- Thanks to $\sum_{c\in\mathcal{C}(p)} \boldsymbol{F}_{pc} = \boldsymbol{0}$, we finally have an explicit expression of $\boldsymbol{U}_p$

$$\Big(\sum_{c\in\mathcal{C}(p)} \mathsf{M}_{pc}\Big)\, \boldsymbol{U}_p = \sum_{c\in\mathcal{C}(p)} \left[P_h^c(\boldsymbol{X}_p,t)\, l_{pc}\boldsymbol{n}_{pc} + \mathsf{M}_{pc}\, \boldsymbol{U}_h^c(\boldsymbol{X}_p,t)\right]$$

  where $\mathsf{M}_{pc} = Z_c\,(l_{pc}^{+}\boldsymbol{n}_{pc}^{+}\otimes\boldsymbol{n}_{pc}^{+} + l_{pc}^{-}\boldsymbol{n}_{pc}^{-}\otimes\boldsymbol{n}_{pc}^{-})$ are positive semi-definite matrices with a physical dimension of a density times a velocity.

Midpoint solver

- The use of the condition $\boldsymbol{F}_{mL} + \boldsymbol{F}_{mR} = \boldsymbol{0}$ leads to

$$\mathrm{M}_m\, \boldsymbol{U}_m = \mathrm{M}_m \left(\frac{Z_L\, \boldsymbol{U}_h^L(\boldsymbol{X}_m) + Z_R\, \boldsymbol{U}_h^R(\boldsymbol{X}_m)}{Z_L + Z_R} \right) - \frac{P_h^R(\boldsymbol{X}_m) - P_h^L(\boldsymbol{X}_m)}{Z_L + Z_R}\; l_{pp^+}\, \boldsymbol{n}_{pp^+},$$

where the matrix $\mathrm{M}_m = \frac{1}{Z_L}\mathrm{M}_{mL} = \frac{1}{Z_R}\mathrm{M}_{mR}$ writes $\mathrm{M}_m = l_{pp^+}\, \boldsymbol{n}_{pp^+} \otimes \boldsymbol{n}_{pp^+}$

Approximate Riemann problem solution

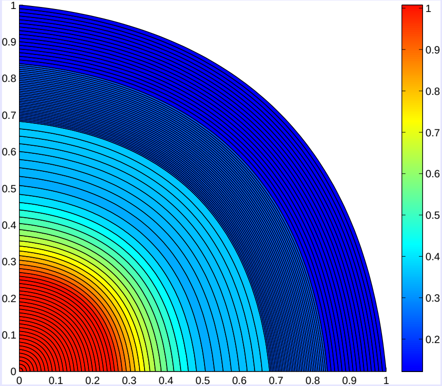
- $$(\boldsymbol{U}_m \cdot \boldsymbol{n}_{pp^+}) = \left(\frac{Z_L\, \boldsymbol{U}_h^L(\boldsymbol{X}_m) + Z_R\, \boldsymbol{U}_h^R(\boldsymbol{X}_m)}{Z_L + Z_R} \right) \cdot \boldsymbol{n}_{pp^+} - \frac{P_h^R(\boldsymbol{X}_m) - P_h^L(\boldsymbol{X}_m)}{Z_L + Z_R}$$

- Regarding the tangential contribution, we make the choice of

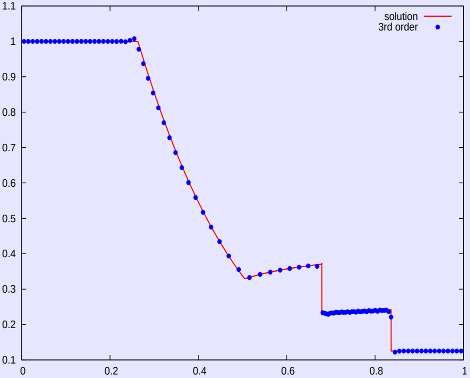
$$(\boldsymbol{U}_m \cdot \boldsymbol{t}_{pp^+}) = \left(\frac{Z_L\, \boldsymbol{U}_h^L(\boldsymbol{X}_m) + Z_R\, \boldsymbol{U}_h^R(\boldsymbol{X}_m)}{Z_L + Z_R} \right) \cdot \boldsymbol{t}_{pp^+}$$

Numerical results

Polar Sod shock tube problem



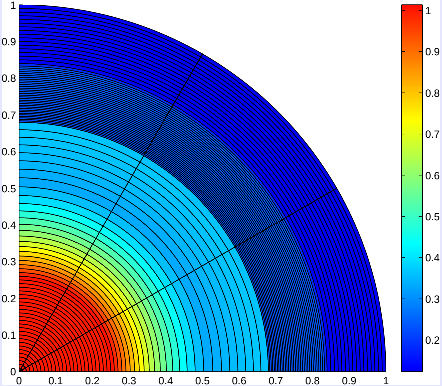
(a) Density map.



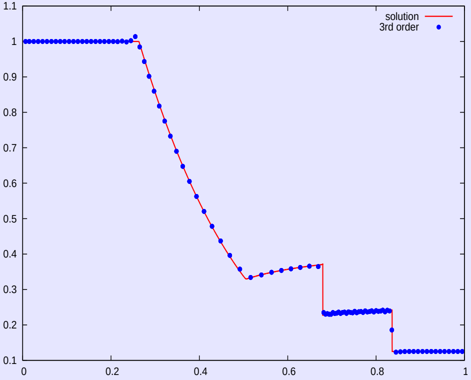
(b) Density profil.

FIG.: Sod shock tube problem on a polar grid made of 100 $\times$ 1 cells.

Polar Sod shock tube problem



(a) Density map.

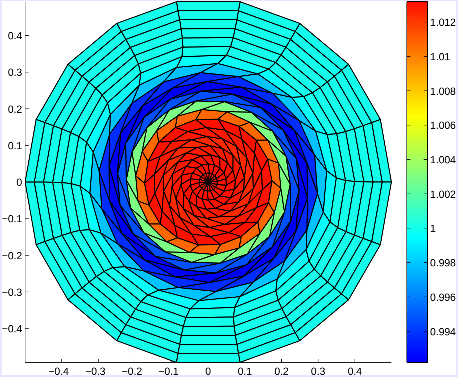


(b) Density profil.

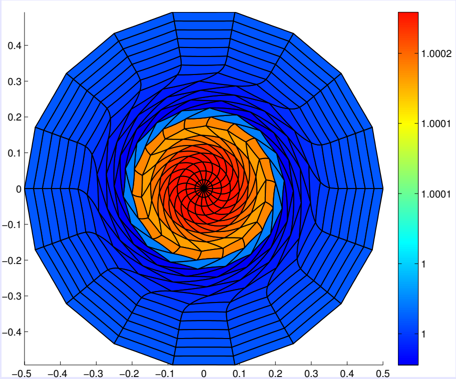
FIG.: Sod shock tube problem on a polar grid made of 100 $\times$ 3 cells.

Numerical results

Variant of the Gresho vortex problem



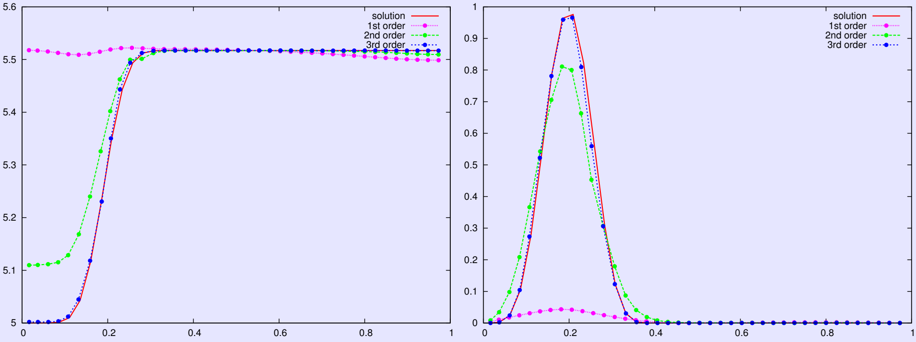
(a) Second-order scheme.



(b) Third-order.

FIG.: Gresho vairant problem on a polar grid defined in polar coordinates by $(r, \theta) \in [0, 1] \times [0, 2\pi]$, with 40×18 cells at $t = 0.36$: zoom of the desity map on the zone $(r, \theta) \in [0, 0.5] \times [0, 2\pi]$.

Variant of the Gresho vortex problem

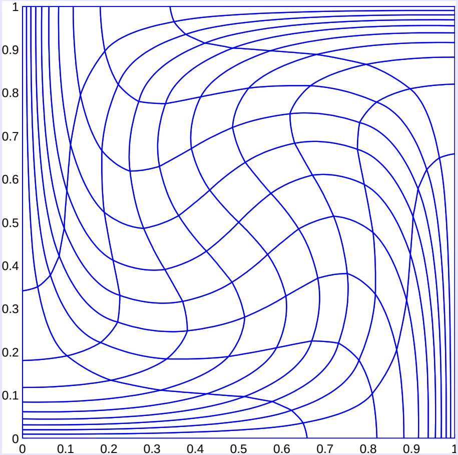


(a) Pressure profil. (b) Velocity profil.

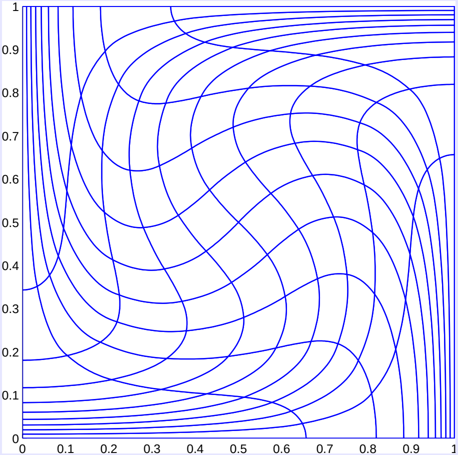
FIG.: Gresho vairant problem on a polar grid defined in polar coordinates by $(r, \theta) \in [0, 1] \times [0, 2\pi]$, with 40×18 cells at $t = 0.36$.

Numerical results

Taylor-Green vortex problem



(a) Third-order scheme.



(b) Exact solution.

FIG.: Motion of a 10×10 Cartesian mesh through a T.-G. vortex, at $t = 0.75$.

Rate of convergence computed on the pressure in the case of the Taylor-Green vortex

	L_1		L_2		L_∞	
h	$E^h_{L_1}$	$q^h_{L_1}$	$E^h_{L_2}$	$q^h_{L_2}$	$E^h_{L_\infty}$	$q^h_{L_\infty}$
$\frac{1}{10}$	4.39E-3	3.00	7.73E-3	2.68	3.90E-2	1.93
$\frac{1}{20}$	5.50E-4	3.04	1.21E-3	3.10	1.03E-2	2.98
$\frac{1}{40}$	6.68E-5	2.91	1.40E-4	2.87	1.30E-3	2.66
$\frac{1}{80}$	8.90E-6	2.89	1.92E-5	2.83	2.11E-4	2.74
$\frac{1}{160}$	1.20E-6	-	2.70E-6	-	3.16E-5	-

TAB.: Third-order DG scheme without limitation at time $t = 0.6$.

	L_1		L_2		L_∞	
h	$E^h_{L_1}$	$q^h_{L_1}$	$E^h_{L_2}$	$q^h_{L_2}$	$E^h_{L_\infty}$	$q^h_{L_\infty}$
$\frac{1}{10}$	2.67E-4	2.96	3.36E-7	2.94	1.21E-3	2.86
$\frac{1}{20}$	3.43E-5	2.97	4.36E-5	2.96	1.66E-4	2.93
$\frac{1}{40}$	4.37E-6	2.99	5.59E-6	2.98	2.18E-5	2.96
$\frac{1}{80}$	5.50E-7	2.99	7.06E-7	2.99	2.80E-6	2.99
$\frac{1}{160}$	6.91E-8	-	8.87E-8	-	3.53E-7	-

TAB.: Third-order DG scheme with limitation at time $t = 0.1$.

Conclusions and perspectives

Conclusions

- We developped a 2nd and a 3rd order DG scheme for the 2D gas dynamics system in Lagrangian formalism with particular geometric consideration
- Numerical fluxes study
- Riemann invariants limitation
- GCL and Piola compatibility condition ensured by construction

Prospects

- High-order limitation on curved geometries
- Implementation of a 3rd order DG scheme on moving mesh
- Extension to ALE