Positivity-preserving cell-centered Lagrangian schemes

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1. Cell-Centered Lagrangian schemes
2. Lagrangian and Eulerian descriptions
3. Compatible first-order positivity-preserving discretization
4. High-order positivity-preserving extension
5. Numerical results
6. Conclusion

Finite volume schemes on moving mesh

- J. K. Dukowicz: CAVEAT scheme, 1986
- B. Després: GLACE scheme, 2005
- P.-H. Maire: EUCCLHYD scheme, 2007
- J. Cheng: High-order ENO conservative Lagrangian scheme, 2007
- G. Kluth: Cell-centered Lagrangian scheme for the hyperelasticity, 2010
- S. Del Pino: Curvilinear finite-volume Lagrangian scheme, 2010
- P. Hoch: Finite volume method on unstructured conical meshes, 2011

DG scheme on initial mesh

- **R. Loubère: DG scheme for Lagrangian hydrodynamics, 2004**
- Z. Jia: DG spectral finite element for Lagrangian hydrodynamics, 2010
- F. Vilar: High-order DG scheme for Lagrangian hydrodynamics, 2012

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Flow transformation of the fluid

- The fluid flow is described mathematically by the continuous transformation, Φ, so-called mapping such as $\Phi : \boldsymbol{X} \longrightarrow \boldsymbol{x} = \Phi(\boldsymbol{X}, t)$

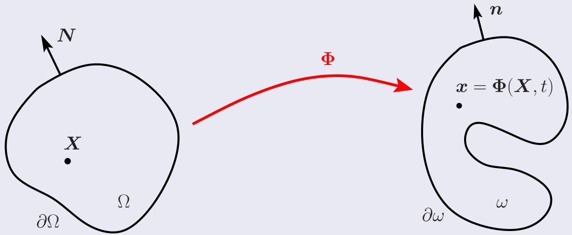

Figure: Notation for the flow map.

where $\boldsymbol{X}$ is the Lagrangian (initial) coordinate, $\boldsymbol{x}$ the Eulerian (actual) coordinate, $\boldsymbol{N}$ the Lagrangian normal and $\boldsymbol{n}$ the Eulerian normal

Deformation Jacobian matrix: deformation gradient tensor

- $\mathrm{F} = \nabla_X \Phi = \frac{\partial \boldsymbol{x}}{\partial X}$ and $J = \det \mathrm{F} > 0$

Trajectory equation

- $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{U}(\boldsymbol{x},t), \quad \boldsymbol{x}(\boldsymbol{X},0) = \boldsymbol{X}$

Material time derivative

- $\frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{x},t) = \frac{\partial}{\partial t} f(\boldsymbol{x},t) + \boldsymbol{U} \cdot \nabla_x f(\boldsymbol{x},t)$

Transformation formulas

$\mathsf{F}\mathrm{d}\boldsymbol{X} = \mathrm{d}\boldsymbol{x}$	Change of shape of infinitesimal vectors
$\rho^0 = \rho J$	Mass conservation
$J\mathrm{d}V = \mathrm{d}v$	Measure of the volume change
$J\mathsf{F}^{-t}\boldsymbol{N}\mathrm{d}S = \boldsymbol{n}\mathrm{d}s$	**Nanson formula**

Differential operators transformations

$\nabla_x P = \frac{1}{J}\nabla_X \cdot (P J \mathsf{F}^{-\mathrm{t}})$	Gradient operator
$\nabla_x \cdot \boldsymbol{U} = \frac{1}{J}\nabla_X \cdot (J\mathsf{F}^{-1}\boldsymbol{U})$	Divergence operator

Piola compatibility condition

- $\nabla_x \centerdot (J\mathrm{F}^{-t}) = \mathbf{0} \quad \Longrightarrow \quad \int_\Omega \nabla_x \centerdot (J\mathrm{F}^{-t})\,\mathrm{d}V = \int_{\partial\Omega} J\mathrm{F}^{-t}\,\boldsymbol{N}\,\mathrm{d}S = \int_{\partial\omega} \boldsymbol{n}\,\mathrm{d}s = \mathbf{0}$

Deformation gradient tensor

- $\dfrac{\mathrm{d}\,\mathrm{F}}{\mathrm{d}t} - \nabla_X \boldsymbol{U} = 0$

Actual configuration

- $\rho \dfrac{\mathrm{d}}{\mathrm{d}t}(\dfrac{1}{\rho}) - \nabla_x \centerdot \boldsymbol{U} = 0$
- $\rho \dfrac{\mathrm{d}\,\boldsymbol{U}}{\mathrm{d}t} + \nabla_x P = \mathbf{0}$
- $\rho \dfrac{\mathrm{d}\,e}{\mathrm{d}t} + \nabla_x \centerdot (P\boldsymbol{U}) = 0$

Initial configuration

- $\rho^0 \dfrac{\mathrm{d}}{\mathrm{d}t}(\dfrac{1}{\rho}) - \nabla_X \centerdot (J\mathrm{F}^{-1}\boldsymbol{U}) = 0$
- $\rho^0 \dfrac{\mathrm{d}\,\boldsymbol{U}}{\mathrm{d}t} + \nabla_X \centerdot (P\,J\mathrm{F}^{-t}) = \mathbf{0}$
- $\rho^0 \dfrac{\mathrm{d}\,e}{\mathrm{d}t} + \nabla_X \centerdot (J\mathrm{F}^{-1}P\boldsymbol{U}) = 0$

Specific internal energy

- $\varepsilon = e - \frac{1}{2}\boldsymbol{U}^2$

Ideal EOS for the perfect gas

- $P = \rho\,(\gamma - 1)\,\varepsilon$ where $a = \sqrt{\frac{\gamma\,P}{\rho}}$

Stiffened EOS for water

- $P = \rho\,(\gamma - 1)\,\varepsilon - \gamma\,P^{\star}$ where $a = \sqrt{\frac{\gamma\,(P+P^{\star})}{\rho}}$

Jones-Wilkins-Lee (JWL) EOS for the detonation-products gas

- $P = \rho\,(\gamma - 1)\,\varepsilon + f(\rho)$ where $a = \sqrt{\frac{\gamma\,P - f(\rho) + \rho\,f'(\rho)}{\rho}}$

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Mass averaged values equations

- $m_c(\frac{1}{\rho})_c^{n+1} = m_c(\frac{1}{\rho})_c^n + \Delta t \sum_{p \in \mathcal{Q}(\partial\omega_c)} \boldsymbol{U}_p^n \,\boldsymbol{.}\, l_{pc}^n \boldsymbol{n}_{pc}^n$
- $m_c \boldsymbol{U}_c^{n+1} = m_c \boldsymbol{U}_c^n - \Delta t \sum_{p \in \mathcal{Q}(\partial\omega_c)} \boldsymbol{F}_{pc}^n$
- $m_c e_c^{n+1} = m_c e_c^n - \Delta t \sum_{p \in \mathcal{Q}(\partial\omega_c)} \boldsymbol{U}_p^n \,\boldsymbol{.}\, \boldsymbol{F}_{pc}^n$

Definitions

- $\psi_c = \frac{1}{m_c} \int_{\Omega_c} \rho^0 \, \psi \, \mathrm{d}V = \frac{1}{m_c} \int_{\omega_c} \rho \, \psi \, \mathrm{d}v$ mean value
- $\boldsymbol{F}_{pc} = P_c \, l_{pc} \boldsymbol{n}_{pc} - \mathrm{M}_{pc}(\boldsymbol{U}_p - \boldsymbol{U}_c)$ subcell forces

Momentum and total energy conservation

- $\sum_{c \in \mathcal{C}(p)} \boldsymbol{F}_{pc} = \boldsymbol{0} \quad \Longrightarrow \quad (\sum_{c \in \mathcal{C}(p)} \mathrm{M}_{pc}) \boldsymbol{U}_p = \sum_{c \in \mathcal{C}(p)} (P_c \, l_{pc} \boldsymbol{n}_{pc} + \mathrm{M}_{pc} \boldsymbol{U}_c)$

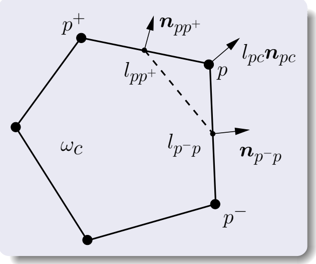
GLACE assumptions

a) $\mathcal{Q}(\partial\omega_c) = \mathcal{P}(\omega_c)$ the node set

b) $l_{pc}\boldsymbol{n}_{pc} = l_{pc}^{-}\boldsymbol{n}_{pc}^{-} + l_{pc}^{+}\boldsymbol{n}_{pc}^{+} = \frac{1}{2}\, l_{p^-p}\boldsymbol{n}_{p^-p} + \frac{1}{2}\, l_{pp^+}\boldsymbol{n}_{pp^+}$

c) $\mathrm{M}_{pc} = Z_{pc}\, l_{pc}\boldsymbol{n}_{pc} \otimes \boldsymbol{n}_{pc}$

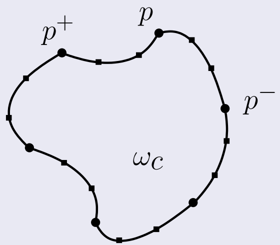
d) $\boldsymbol{U}_p = \Big(\sum_{c\in\mathcal{C}(p)} \mathrm{M}_{pc}\Big)^{-1} \sum_{c\in\mathcal{C}(p)} (P_c\, l_{pc}\boldsymbol{n}_{pc} + \mathrm{M}_{pc}\boldsymbol{U}_c)$

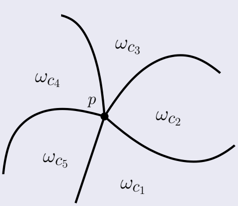


EUCCLHYD assumptions

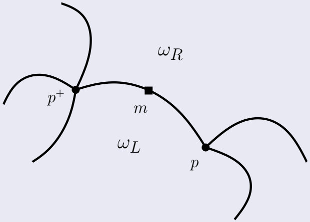
- Same assumptions a), b) and d) as GLACE

c) $\mathrm{M}_{pc} = Z_{pc}^{-}\, l_{pc}^{-}\boldsymbol{n}_{pc}^{-} \otimes \boldsymbol{n}_{pc}^{-} + Z_{pc}^{+}\, l_{pc}^{+}\boldsymbol{n}_{pc}^{+} \otimes \boldsymbol{n}_{pc}^{+}$





Node cell set



Middle point cell set

Cell-centered DG (CCDG) assumptions

a) $\mathcal{Q}(\partial\omega_c) = \bigcup_{p\in\mathcal{P}(\omega_c)} (\mathcal{Q}(pp^+) \setminus \{p^+\})$

b) For $q \in \mathcal{Q}(pp^+)$, $\quad l_q \boldsymbol{n}_{q|_{pp^+}} = \int_0^1 \lambda_q(\zeta) \sum_{k\in\mathcal{Q}(pp^+)} \frac{\partial\lambda_k}{\partial\zeta}(\boldsymbol{x}_k \times \boldsymbol{e}_z)\,\mathrm{d}\zeta$

For $p \in \mathcal{P}(\omega_c)$, $\quad l_{pc}\boldsymbol{n}_{pc} = l_p\boldsymbol{n}_{p|_{p^-p}} + l_p\boldsymbol{n}_{p|_{pp^+}}$

For $q \in \mathcal{Q}(pp^+) \setminus \{p, p^+\}$, $\quad l_{qc}\boldsymbol{n}_{qc} = l_q\boldsymbol{n}_{q|_{pp^+}}$

CCDG assumptions

c) For $p \in \mathcal{P}(\omega_c)$, $\quad \mathsf{M}_{pc} = Z_{pc}^- \, l_{pc}^- \boldsymbol{n}_{pc}^- \otimes \boldsymbol{n}_{pc}^- + Z_{pc}^+ \, l_{pc}^+ \boldsymbol{n}_{pc}^+ \otimes \boldsymbol{n}_{pc}^+$

For $q \in \mathcal{Q}(pp^+) \setminus \{p, p^+\}$, $\quad \mathsf{M}_{pc} = Z_{pc} \, l_{pc} \boldsymbol{n}_{pc} \otimes \boldsymbol{n}_{pc}$

d) For $p \in \mathcal{P}(\omega_c)$, $\quad \boldsymbol{U}_p = \Big(\sum_{c \in \mathcal{C}(p)} \mathsf{M}_{pc} \Big)^{-1} \sum_{c \in \mathcal{C}(p)} (P_c \, l_{pc} \boldsymbol{n}_{pc} + \mathsf{M}_{pc} \boldsymbol{U}_c)$

For $q \in \mathcal{Q}(pp^+) \setminus \{p, p^+\}$, $\quad \boldsymbol{U}_p = \dfrac{Z_{pL} \, \boldsymbol{U}_L + Z_{pR} \, \boldsymbol{U}_R}{Z_{pL} + Z_{pR}} - \dfrac{P_R - P_L}{Z_{pL} + Z_{pR}} \, \boldsymbol{n}_{pL}$

CFL condition

- System eigenvalues: $-a,\ 0,\ a$

$$\forall c, \quad \Delta t \leq C_e \, \frac{v_c^n}{a_c \, L_c}$$

Volume control

- Relative volume variation: $\dfrac{|v_c^{n+1} - v_c^n|}{v_c^n} \leq C_v$

$$\forall c, \quad \Delta t \leq C_v \, \frac{v_c^n}{|\displaystyle\sum_{p \in \mathcal{Q}(\partial\omega_c)} \boldsymbol{U}_p^n \cdot l_{pc}^n \boldsymbol{n}_{pc}^n|}$$

Admissible set

- $\mathrm{W} = (\frac{1}{\rho}, \boldsymbol{U}, e)^{\mathrm{t}}$
- $G = \{\mathrm{W}, \quad \rho > 0,\ \varepsilon = e - \frac{1}{2}\boldsymbol{U}^2 > 0,\ a^2 = (\partial_\rho P)_{|s} > 0\}$

Ideal EOS

- If $\rho > 0$ then $\varepsilon > 0 \iff a^2 = \gamma(\gamma - 1)\varepsilon > 0 \iff P = \rho(\gamma - 1)\varepsilon > 0$
- $G = \{\mathrm{W}, \quad \rho > 0 \text{ and } \varepsilon = e - \frac{1}{2}\boldsymbol{U}^2 > 0\}$ convex set

First-order positivity-preserving scheme

- If $\mathrm{W}_c^n = ((\frac{1}{\rho})_c^n, \boldsymbol{U}_c^n, e_c^n)^{\mathrm{t}} \in G$, then under which constraint $\mathrm{W}_c^{n+1} \in G$?

Positive density

- If $(\frac{1}{\rho})_c^n > 0$ then $(\frac{1}{\rho})_c^{n+1} > 0 \iff (\frac{1}{\rho})_c^n > -\dfrac{\Delta t}{m_c} \displaystyle\sum_{p \in \mathcal{Q}(\partial\omega_c)} \boldsymbol{U}_p^n \cdot l_{pc}^n \boldsymbol{n}_{pc}^n$
- Thus if $C_v < 1$ then $(\frac{1}{\rho})_c^n = \frac{v_c^n}{m_c} > 0 \implies (\frac{1}{\rho})_c^{n+1} = \frac{v_c^{n+1}}{m_c} > 0$

Positive internal energy and pressure

- $\varepsilon_c = e_c - \frac{1}{2}(\boldsymbol{U}_c)^2$
- $\varepsilon_c^{n+1} = \varepsilon_c^n - \dfrac{\Delta t}{m_c}\left(\sum_p \boldsymbol{U}_p^n \cdot \boldsymbol{F}_{pc}^n - \sum_p \boldsymbol{U}_c^n \cdot \boldsymbol{F}_{pc}^n + \dfrac{\Delta t}{2 m_c}\Big(\sum_p \boldsymbol{F}_{pc}^n\Big)^2\right)$

Properties

- $\boldsymbol{F}_{pc} = P_c\, l_{pc} \boldsymbol{n}_{pc} - \mathsf{M}_{pc}(\boldsymbol{U}_p - \boldsymbol{U}_c)$
- $\displaystyle\sum_{p \in \mathcal{Q}(\partial\omega_c)} l_{pc} \boldsymbol{n}_{pc} = \sum_{p \in \mathcal{P}(\omega_c)} l_{pp^+} \boldsymbol{n}_{pp^+} = \boldsymbol{0}$

Definitions

- $\lambda_c = \dfrac{\Delta t}{m_c}$ and $\boldsymbol{V}_p = \boldsymbol{U}_p^n - \boldsymbol{U}_c^n$
- $A_c = \varepsilon_c^n - P_c^n\, \lambda_c \displaystyle\sum_p \boldsymbol{U}_p^n \cdot l_{pc}^n \boldsymbol{n}_{pc}^n$
- $B_c = \displaystyle\sum_p \mathsf{M}_{pc} \boldsymbol{V}_p \cdot \boldsymbol{V}_p - \frac{\lambda_c}{2}\Big(\sum_p \mathsf{M}_{pc} \boldsymbol{V}_p\Big)^2$

Positive pressure and internal energy

- $\varepsilon_c^{n+1} = A_c + \lambda_c B_c$
- Thus if $A_c > 0$ and $B_c \geq 0$ then $\varepsilon_c^{n+1} > 0$

$A_c > 0$

- $A_c = \varepsilon_c^n - \dfrac{P_c^n}{\rho_c^n} \dfrac{v_c^{n+1} - v_c^n}{v_c^n}$
- Thus if $C_v < \dfrac{\rho_c^n \, \varepsilon_c^n}{P_c^n} = \dfrac{1}{\gamma - 1}$ then $\varepsilon_c^n > 0 \implies A_c > 0$

Entropy

- $T\mathrm{d}S = \mathrm{d}\varepsilon + P\mathrm{d}(\frac{1}{\rho}) \geq 0$ Gibbs identity + second law of thermodynamics

Discrete entropy inequality

- $\lambda_c B_c = \varepsilon_c^{n+1} - A_c = \varepsilon_c^{n+1} - \varepsilon_c^n + P_c^n \left((\frac{1}{\rho})_c^{n+1} - (\frac{1}{\rho})_c^n \right)$

$B_c \geq 0$

- $B_c = \sum_{p \in \mathcal{Q}(\partial\omega_c)} \mathrm{M}_{pc} \boldsymbol{V}_p \cdot \boldsymbol{V}_p - \frac{\lambda_c}{2} \left(\sum_{p \in \mathcal{Q}(\partial\omega_c)} \mathrm{M}_{pc} \boldsymbol{V}_p \right)^2$
- $\mathrm{M}_{pc} = \sum_{n=1}^{N_p} Z_{p_n} l_{p_n} \boldsymbol{n}_{p_n} \otimes \boldsymbol{n}_{p_n}$
- $\sum_{p \in \mathcal{Q}(\partial\omega_c)} \mathrm{M}_{pc} \boldsymbol{V}_p \cdot \boldsymbol{V}_p = \sum_{p \in \mathcal{Q}(\partial\omega_c)} \sum_{n=1}^{N_p} Z_{p_n} l_{p_n} (\boldsymbol{V}_p \cdot \boldsymbol{n}_{p_n})^2 = \sum_{p \in \mathcal{Q}(\partial\omega_c)} \sum_{n=1}^{N_p} Z_{p_n} l_{p_n} X_{p_n}^2$
- Re-numbering: $\sum_{p \in \mathcal{Q}(\partial\omega_c)} \sum_{n=1}^{N_p} \psi_{p_n} = \sum_{i=1}^{N_c} \psi_i$
- $B_c = \sum_{i=1}^{N_c} Z_i l_i X_i^2 - \frac{\lambda_c}{2} \sum_{i,j=1}^{N_c} Z_i Z_j l_i l_j X_i X_j (\boldsymbol{n}_i \cdot \boldsymbol{n}_j) = \mathrm{H} \boldsymbol{X} \cdot \boldsymbol{X}$,

where $\boldsymbol{X} = (X_1, \ldots, X_{N_c})^{\mathrm{t}}$ and $\mathrm{H}_{ij} = \begin{cases} Z_i l_i (1 - \frac{\lambda_c}{2} Z_i l_i), & \text{if } i = j, \\ -\frac{\lambda_c}{2} Z_i Z_j l_i l_j (\boldsymbol{n}_i \cdot \boldsymbol{n}_j), & \text{if } i \neq j. \end{cases}$

Theorem

- If H is symmetric diagonally dominant with non-negative diagonal entries then H is positive semi-definite (thanks to Gerschgorin theorem)

$B_c \geq 0$

- If $\lambda_c \leq \dfrac{2}{Z_i\, l_i}$ then $\mathrm{H}_{ii} \geq 0$
- If $\lambda_c \leq \dfrac{2}{\sum\limits_j Z_j\, l_j |\boldsymbol{n}_i \,.\, \boldsymbol{n}_j|}$ then $|\mathrm{H}_{ii}| - \sum\limits_{j \neq i} |\mathrm{H}_{ij}| \geq 0$
- Thus if $\lambda_c \leq \dfrac{2}{\sum\limits_j Z_j\, l_j} \iff \Delta t \leq \dfrac{m_c}{\frac{1}{2} \sum\limits_j Z_j\, l_j}$ then $B_c \geq 0$

Acoustic impedance $Z_c = \rho_c\, a_c$

- If $\Delta t \leq \dfrac{v_c^n}{a_c\, L_c}$ where $L_c = \frac{1}{2} \sum_j l_j$ then $B_c \geq 0$

Positivity-preserving property

Finally, for the first-order finite volume cell-centered Lagrangian schemes, if

1. $\mathrm{W}_c^n \in G$
2. $$\Delta t \leq C_v \frac{v_c^n}{|\sum\limits_{p \in \mathcal{Q}(\partial\omega_c)} \boldsymbol{U}_p^n \,.\, l_{pc}^n \boldsymbol{n}_{pc}^n|}, \qquad \text{with} \quad C_v < \min\left(1, \frac{1}{\gamma - 1}\right)$$
3. $$\Delta t \leq \frac{v_c^n}{a_c\, L_c}, \qquad \text{with} \quad L_c = \begin{cases} \frac{1}{2} \sum\limits_{p \in \mathcal{P}(\omega_c)} l_{pc}, & \text{GLACE} \\ \frac{1}{2} \sum\limits_{p \in \mathcal{P}(\omega_c)} l_{pp^+}, & \text{EUCCLHYD} \\ \frac{1}{2} \sum\limits_{p \in \mathcal{P}(\omega_c)} \sum\limits_{q \in \mathcal{Q}(pp^+)} l_{q|_{pp^+}}. & \text{CCDG} \end{cases}$$

Then $\quad \mathrm{W}_c^{n+1} \in G \quad$ and $\quad \varepsilon_c^{n+1} - \varepsilon_c^n + P_c^n \left((\frac{1}{\rho})_c^{n+1} - (\frac{1}{\rho})_c^n \right) \geq 0$

Norm definitions

- $\|\psi\|_{L_1} = \int_\Omega \rho^0 \, |\psi| \, \mathrm{d}V = \int_\omega \rho \, |\psi| \, \mathrm{d}v$
- $\|\psi\|_{L_2} = \left(\int_\Omega \rho^0 \, \psi^2 \, \mathrm{d}V \right)^{\frac{1}{2}} = \left(\int_\omega \rho \, \psi^2 \, \mathrm{d}v \right)^{\frac{1}{2}}$

Stability analysis

- For sake of simplicity periodic boundary conditions (PBC) are considered
- ψ_h^n is the piecewise constant numerical solution such as $\psi^n_{h|_{\omega_c}} = \psi_c^n$
- We assume the initial solution vector $\mathrm{W}_c^0 = ((\frac{1}{\rho})_c^0, \boldsymbol{U}_c^0, e_c^0)^{\mathrm{t}}$ on cell ω_c is computed through

$$\mathrm{W}_c^0 = \frac{1}{m_c} \int_{\Omega_c} \rho^0(\boldsymbol{X}) \, \mathrm{W}^0(\boldsymbol{X}) \, \mathrm{d}V,$$

where $\mathrm{W}^0 = (\frac{1}{\rho^0}, \boldsymbol{U}^0, e^0)^{\mathrm{t}}$ and $\frac{1}{\rho^0}, \boldsymbol{U}^0, e^0$ respectively are the initial specific volume, velocity and total energy

Specific volume

- Positivity $|(\frac{1}{\rho})_c^n| = (\frac{1}{\rho})_c^n$
- Conservation $\sum_c m_c\,(\frac{1}{\rho})_c^n = \sum_c m_c\,(\frac{1}{\rho})_c^{n-1}$ (since PBC + $\sum_{c\in\mathcal{C}(p)} l_{pc}\boldsymbol{n}_{pc} = \boldsymbol{0}$)

$$\|(\frac{1}{\rho})_h^n\|_{L_1} = \sum_c m_c\,|(\frac{1}{\rho})_c^n| = \sum_c m_c\,|(\frac{1}{\rho})_c^{n-1}| = \|(\frac{1}{\rho})_h^{n-1}\|_{L_1}$$

Total energy

- Positivity $|e_c^n| = e_c^n$ (since $\varepsilon_c^n > 0 \Longleftrightarrow e_c^n > \frac{1}{2}(\boldsymbol{U}_c^n)^2 \geq 0$)
- Conservation $\sum_c m_c\, e_c^n = \sum_c m_c\, e_c^{n-1}$ (since PBC + $\sum_{c\in\mathcal{C}(p)} \boldsymbol{F}_{pc} = \boldsymbol{0}$)

$$\|e_h^n\|_{L_1} = \sum_c m_c\,|e_c^n| = \sum_c m_c\,|e_c^{n-1}| = \|e_h^{n-1}\|_{L_1}$$

Kinetic energy and velocity

- $K = \frac{1}{2}\boldsymbol{U}^2$ specific kinetic energy
- $\frac{1}{2}(\boldsymbol{U}_c^n)^2 < e_c^n \quad \Longrightarrow \quad \frac{1}{2}\sum_c m_c\,(\boldsymbol{U}_c^n)^2 < \sum_c m_c\, e_c^n$
- $2m_c e_c^n = 2\sqrt{m_c}\,\sqrt{m_c\,(e_c^n)^2} \leq m_c + m_c\,(e_c^n)^2$
- $\sum_c m_c\,(\boldsymbol{U}_c^n)^2 < \sum_c m_c + \sum_c m_c\,(e_c^n)^2$

Stability

- $\|(\frac{1}{\rho})_h^n\|_{L_1} = \|\frac{1}{\rho^0}\|_{L_1}$
- $\|e_h^n\|_{L_1} = \|e^0\|_{L_1}$
- $\|K_h^n\|_{L_1} < \|e_h^n\|_{L_1}$
- $\|\boldsymbol{U}_h^n\|_{L_2}^2 < m_\omega + \|e_h^n\|_{L_2}^2$

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Control point solvers

- In the control point solvers, $\boldsymbol{F}_{pc}$ and $\boldsymbol{U}_p$, the **interpolation values** at point p of the high-order approximations of the pressure and velocity, $P_h^c(p)$ and $\boldsymbol{U}_h^c(p)$, are used instead of the mean values P_c and $\boldsymbol{U}_c$

High-order extension

1.
   - Piecewise linear approximations of the pressure and velocity, $P_h(p)$ and $\boldsymbol{U}_h(p)$, are constructed using the mean values, P_c and $\boldsymbol{U}_c$, over the cells (GLACE and EUCCLHYD)
2.
   - A piecewise polynomial reconstruction of the solution vector $\mathrm{W}_h(\boldsymbol{x}) = ((\frac{1}{\rho})_h(\boldsymbol{x}), \boldsymbol{U}_h(\boldsymbol{x}), e_h(\boldsymbol{x}))^{\mathrm{t}}$ is assumed, such as its mass averaged value over cell ω_c corresponds to W_c (CCDG)
   - The pressure is pointwisely defined through the EOS, such as

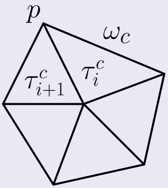
$$P_h(\boldsymbol{x}) = \rho_h(\boldsymbol{x})\,(\gamma-1)\,(e_h(\boldsymbol{x}) - \frac{1}{2}(\boldsymbol{U}_h(\boldsymbol{x})^2))$$

Quadrature rule over triangles

- Exact for polynomials up to degree $2(d-1)$
- containing the cell boundary control points, *i.e.*, $\mathcal{Q}(\partial\Omega_c) \subset \bigcup_{i=1}^{ntri} \mathcal{R}_{i,c}$
- With positive weights, *i.e.*, $\forall q \in \mathcal{R}_{i,c},\ w_q \geq 0$

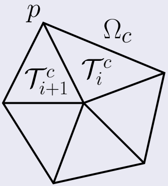
GLACE and EUCCLHYD schemes

- $\psi_c = \dfrac{1}{m_c}\displaystyle\int_{\omega_c} \rho_c\, \psi_h^c \,\mathrm{d}v = \dfrac{1}{m_c}\sum_{i=1}^{ntri} |\tau_i^c| \sum_{q\in\mathcal{R}_{i,c}} w_q\, \rho_c\, \psi_h^c(q)$
- $m_q^c = \displaystyle\sum_{i,\mathcal{R}_{i,c}\ni q} |\tau_i^c|\, w_q\, \rho_c$



CCDG scheme

- $\psi_c = \dfrac{1}{m_c}\displaystyle\int_{\Omega_c} \rho^0\, \psi_h^c \,\mathrm{d}V = \dfrac{1}{m_c}\sum_{i=1}^{ntri} |\mathcal{T}_i^c| \sum_{q\in\mathcal{R}_{i,c}} w_q\, \rho^0(q)\, \psi_h^c(q)$
- $m_q^c = \displaystyle\sum_{i,\mathcal{R}_{i,c}\ni q} |\mathcal{T}_i^c|\, w_q\, \rho^0(q)$



Properties

- $\mathcal{R}_c = \bigcup_{i=1}^{ntri} \mathcal{R}_{i,c}$
- $m_c = \int_{\Omega_c} \rho^0 \,\mathrm{d}V = \rho_c \int_{\omega_c} \mathrm{d}v = \sum_{q \in \mathcal{R}_c} m_q^c$
- $\psi_c = \frac{1}{m_c} \sum_{q \in \mathcal{R}_c} m_q^c \, \psi_h^c(q)$
- $m_\star^c = m_c - \sum_{p \in \mathcal{Q}(\partial\omega_c)} m_p^c$
- $\psi_\star^c = \frac{1}{m_\star^c} \sum_{q \in \mathcal{R}_c \setminus \mathcal{Q}(\partial\omega_c)} m_q^c \, \psi_h^c(q)$
- $\psi_c = \frac{m_\star^c}{m_c} \psi_\star^c + \frac{1}{m_c} \sum_{p \in \mathcal{Q}(\partial\omega_c)} m_p^c \, \psi_h^c(p)$

Mass averaged value equations

- $m_c(\frac{1}{\rho})_c^{n+1} = m_c(\frac{1}{\rho})_c^n + \Delta t \sum_{p\in\mathcal{Q}(\partial\omega_c)} \boldsymbol{U}_p^n \,\boldsymbol{.}\, l_{pc}^n \boldsymbol{n}_{pc}^n$
- $m_c \boldsymbol{U}_c^{n+1} = m_c \boldsymbol{U}_c^n - \Delta t \sum_{p\in\mathcal{Q}(\partial\omega_c)} \boldsymbol{F}_{pc}^n$
- $m_c e_c^{n+1} = m_c e_c^n - \Delta t \sum_{p\in\mathcal{Q}(\partial\omega_c)} \boldsymbol{U}_p^n \,\boldsymbol{.}\, \boldsymbol{F}_{pc}^n$

Decomposition

- $(\frac{1}{\rho})_c^{n+1} = \frac{m_\star^c}{m_c}(\frac{1}{\rho})_\star^c + \frac{1}{m_c} \sum_{p\in\mathcal{Q}(\partial\omega_c)} m_p^c \Big((\frac{1}{\rho})_h^c(p) + \frac{\Delta t}{m_p^c} \boldsymbol{U}_p^n \,\boldsymbol{.}\, l_{pc}^n \boldsymbol{n}_{pc}^n \Big)$
- $\boldsymbol{U}_c^{n+1} = \frac{m_\star^c}{m_c} \boldsymbol{U}_\star^c + \frac{1}{m_c} \sum_{p\in\mathcal{Q}(\partial\omega_c)} m_p^c \Big(\boldsymbol{U}_h^c(p) - \frac{\Delta t}{m_p^c} \boldsymbol{F}_{pc}^n \Big)$
- $e_c^{n+1} = \frac{m_\star^c}{m_c} e_\star^c + \frac{1}{m_c} \sum_{p\in\mathcal{Q}(\partial\omega_c)} m_p^c \Big(e_h^c(p) - \frac{\Delta t}{m_p^c} \boldsymbol{U}_p^n \,\boldsymbol{.}\, \boldsymbol{F}_{pc}^n \Big)$

Procedure

- Express these equations as a convex combination of first-order schemes

X. ZHANG, Y. XIA, C.-W. SHU, *Maximum-principle-satisfying and positivity-preserving high order discontinuous Galerkin schemes for conservation laws on triangular meshes.* J. Sci. Comp., 50:29-62, 2012.

J. CHENG and C.-W. SHU, *Positivity-preserving Lagrangian scheme for multi-material compressible flow.* J. Comp. Phys., 257:143-168, 2014.

Specific volume

- $$\sum_{p\in\mathcal{Q}(\partial\omega_c)} l_{pc}\boldsymbol{n}_{pc} = \boldsymbol{0} \quad\Longleftrightarrow\quad l_{pc}\boldsymbol{n}_{pc} = -\sum_{q\in\mathcal{Q}(\partial\omega_c)\backslash p} l_{qc}\boldsymbol{n}_{qc}$$
- $$h_p^\rho = (\frac{1}{\rho})_h^c(p) + \frac{\Delta t}{m_p^c}\boldsymbol{U}_p^n \,\boldsymbol{.}\, l_{pc}^n\boldsymbol{n}_{pc}^n$$
- $$H_p^\rho = (\frac{1}{\rho})_h^c(p) + \frac{\Delta t}{m_p^c}(\boldsymbol{U}_p^n - \boldsymbol{V}_c)\,\boldsymbol{.}\, l_{pc}^n\boldsymbol{n}_{pc}^n = (\frac{1}{\rho})_h^c(p) + \frac{\Delta t}{m_p^c}\sum_{q\in\mathcal{Q}(\partial\omega_c)} \boldsymbol{V}_q^p \,\boldsymbol{.}\, l_{qc}^n\boldsymbol{n}_{qc}^n$$

where $$\boldsymbol{V}_q^p = \begin{cases} \boldsymbol{U}_p^n, & \text{if } p = q, \\ \boldsymbol{V}_c, & \text{if } p \neq q. \end{cases}$$

Momentum

- $\boldsymbol{h}_p^u = \boldsymbol{U}_h^c(p) - \dfrac{\Delta t}{m_p^c}\,\boldsymbol{F}_{pc}^n$
- $\displaystyle\sum_{p \in \mathcal{Q}(\partial\omega_c)} \mathfrak{F}_{pc} = \mathbf{0} \quad \Longleftrightarrow \quad \mathfrak{F}_{pc} = -\sum_{q \in \mathcal{Q}(\partial\omega_c)\backslash p} \mathfrak{F}_{qc}$
- $\displaystyle\boldsymbol{H}_p^u = \boldsymbol{U}_h^c(p) - \frac{\Delta t}{m_p^c}(\boldsymbol{F}_{pc}^n - \mathfrak{F}_{pc}) = \boldsymbol{U}_h^c(p) - \frac{\Delta t}{m_p^c} \sum_{q \in \mathcal{Q}(\partial\omega_c)} \mathfrak{F}_q^p$

  where $\quad \mathfrak{F}_q^p = \begin{cases} \boldsymbol{F}_{pc}^n, & \text{if } p = q, \\ \mathfrak{F}_{qc}, & \text{if } p \neq q. \end{cases}$

Total energy

- $h_p^e = e_h^c(p) - \dfrac{\Delta t}{m_p^c}\,\boldsymbol{U}_p^n \boldsymbol{.}\, \boldsymbol{F}_{pc}^n$
- $\displaystyle H_p^e = e_h^c(p) - \frac{\Delta t}{m_p^c}(\boldsymbol{U}_p^n \boldsymbol{.}\, \boldsymbol{F}_{pc}^n - \boldsymbol{V}_c \boldsymbol{.}\, \mathfrak{F}_{pc}) = e_h^c(p) - \frac{\Delta t}{m_p^c} \sum_{q \in \mathcal{Q}(\partial\omega_c)} \boldsymbol{V}_q^p \boldsymbol{.}\, \mathfrak{F}_q^p$

Properties

- $\displaystyle\sum_{p\in\mathcal{Q}(\partial\omega_c)} m_p^c\, h_p^\rho = \sum_{p\in\mathcal{Q}(\partial\omega_c)} m_p^c\, H_p^\rho$
- $\displaystyle\sum_{p\in\mathcal{Q}(\partial\omega_c)} m_p^c\, \boldsymbol{h}_p^u = \sum_{p\in\mathcal{Q}(\partial\omega_c)} m_p^c\, \boldsymbol{H}_p^u$
- $\displaystyle\sum_{p\in\mathcal{Q}(\partial\omega_c)} m_p^c\, h_p^e = \sum_{p\in\mathcal{Q}(\partial\omega_c)} m_p^c\, H_p^e$

Mimic the first-order scheme

1. $\displaystyle\sum_{p\in\mathcal{Q}(\partial\omega_c)} \mathfrak{F}_{pc} = \mathbf{0}$
2. $\displaystyle\sum_{q\in\mathcal{Q}(\partial\omega_c)} \mathfrak{F}_q^p = \sum_{q\in\mathcal{Q}(\partial\omega_c)} P_h^c(p)\, l_{qc}^n \boldsymbol{n}_{qc}^n - \mathsf{M}_{qc}(\boldsymbol{V}_q^p - \boldsymbol{U}_h^c(p))$
3. $\displaystyle\sum_{q\in\mathcal{Q}(\partial\omega_c)} \boldsymbol{V}_q^p \,\boldsymbol{.}\, \mathfrak{F}_q^p = P_h^c(p) \sum_{q\in\mathcal{Q}(\partial\omega_c)} \boldsymbol{V}_q^p \,\boldsymbol{.}\, l_{qc}^n \boldsymbol{n}_{qc}^n - \sum_{q\in\mathcal{Q}(\partial\omega_c)} \boldsymbol{V}_q^p \,\boldsymbol{.}\, \mathsf{M}_{qc}(\boldsymbol{V}_q^p - \boldsymbol{U}_h^c(p))$

Artificial cell velocity and subcell forces

- $\mathfrak{F}_{pc} = P_h^c(p) l_{pc}^n \boldsymbol{n}_{pc}^n + (\mathrm{M}_c - \mathrm{M}_{qc})(\boldsymbol{V}_c - \boldsymbol{U}_h^c(p))$

  where $\mathrm{M}_c = \sum\limits_{p \in \mathcal{Q}(\partial\omega_c)} \mathrm{M}_{pc}$

- $\boldsymbol{V}_c = \dfrac{1}{N_Q - 1} \mathrm{M}_c^{-1} \sum\limits_{q \in \mathcal{Q}(\partial\omega_c)} \left((\mathrm{M}_c - \mathrm{M}_{qc}) \boldsymbol{U}_h^c(q) - P_h^c(q) l_{qc}^n \boldsymbol{n}_{qc}^n \right)$

  where $N_Q = |\mathcal{Q}(\partial\omega_c)| = N_P\,(d-1)$ and $N_P = |\mathcal{P}(\omega_c)|$

Convex combination

- $\mathrm{W}_c^{n+1} = \dfrac{1}{m_c} \Big(m_\star^c \, \mathrm{W}_\star^c + \sum\limits_{p \in \mathcal{Q}(\partial\omega_c)} m_p^c \, \mathrm{H}_p^c \Big),$

  where $\mathrm{H}_p^c = (H_p^\rho, \boldsymbol{H}_p^u, H_p^e)^{\mathrm{t}}$ and $m_c = m_\star^c + \sum\limits_{p \in \mathcal{Q}(\partial\omega_c)} m_p^c$

Positivity-preserving property

Finally, for the high-order cell-centered Lagrangian schemes presented, if

1. $\mathrm{W}_c^n \in G,\ \mathrm{W}_\star^c \in G$ and $\forall p \in \mathcal{Q}(\partial\omega_c),\ \mathrm{W}_h^c(p) \in G$
2. $\Delta t \leq C_v \dfrac{m_p^c\,(\frac{1}{\rho})_h^c(p)}{|(\boldsymbol{U}_p^n - \boldsymbol{V}_c)\,.\,l_{pc}^n\boldsymbol{n}_{pc}^n|}$, with $C_v < \min\left(1, \dfrac{\varepsilon_h^c(p)}{|P_h^c(p)|\,(\frac{1}{\rho})_h^c(p)} = \dfrac{1}{\gamma - 1}\right)$
3. $\Delta t \leq \dfrac{m_p^c}{\frac{1}{2}\sum\limits_j Z_j\, l_j} = \dfrac{m_p^c}{m_c}\,\dfrac{v_c^n}{a_c\, L_c}$

Then $\mathrm{W}_c^{n+1} \in G$

Quantities involved

- $\forall p \in \mathcal{Q}(\partial\omega_c), \quad \mathrm{W}_h^c(p) \in G$
- $\mathrm{W}_\star^c = \dfrac{\displaystyle\sum_{q\in\mathcal{R}_c\setminus\mathcal{Q}(\partial\omega_c)} m_q^c\,\mathrm{W}_h^c(q)}{\displaystyle\sum_{p\in\mathcal{R}_c\setminus\mathcal{Q}(\partial\omega_c)} m_p^c} \in G \quad$ or $\quad \forall q \in \mathcal{R}_c \setminus \mathcal{Q}(\partial\omega_c), \quad \mathrm{W}_h^c(q) \in G$

Positive limitation

- $\widetilde{(\frac{1}{\rho})}_h^c = (\frac{1}{\rho})_c + \theta_\rho\,((\frac{1}{\rho})_h^c - (\frac{1}{\rho})_c)$
- $\widetilde{\boldsymbol{U}}_h^c = \boldsymbol{U}_c + \theta_\varepsilon\,(\boldsymbol{U}_h^c - \boldsymbol{U}_c)$
- $\widetilde{e}_h^c = e_c + \theta_\varepsilon\,(e_h^c - e_c)$

where $\theta_\rho \in [0,1]$ and $\theta_\varepsilon \in [0,1]$

Riemann invariants differentials

- $\mathrm{d}\alpha_t = \mathrm{d}\boldsymbol{U} \,.\, \boldsymbol{t}$
- $\mathrm{d}\alpha_- = \mathrm{d}(\frac{1}{\rho}) - \frac{1}{\rho a}\,\mathrm{d}\boldsymbol{U} \,.\, \boldsymbol{n}$
- $\mathrm{d}\alpha_+ = \mathrm{d}(\frac{1}{\rho}) + \frac{1}{\rho a}\,\mathrm{d}\boldsymbol{U} \,.\, \boldsymbol{n}$
- $\mathrm{d}\alpha_e = \mathrm{d}e - \boldsymbol{U} \,.\, \mathrm{d}\boldsymbol{U} + P\,\mathrm{d}(\frac{1}{\rho})$

Mean value linearization

- $\alpha^c_{t,h} = \boldsymbol{U}^c_h \,.\, \boldsymbol{t}$
- $\alpha^c_{-,h} = (\frac{1}{\rho})^c_h - \frac{1}{Z_c}\,\boldsymbol{U}^c_h \,.\, \boldsymbol{n}$
- $\alpha^c_{+,h} = (\frac{1}{\rho})^c_h + \frac{1}{Z_c}\,\boldsymbol{U}^c_h \,.\, \boldsymbol{n}$
- $\alpha^c_{e,h} = e^c_h - \boldsymbol{U}^c_0 \,.\, \boldsymbol{U}^c_h + P^c_0\,(\frac{1}{\rho})^c_h$

Unit direction ensuring symmetry preservation

- $\boldsymbol{n} = \dfrac{\boldsymbol{U}^c_0}{\|\boldsymbol{U}^c_0\|}$ and $\boldsymbol{t} = \boldsymbol{e}_z \times \dfrac{\boldsymbol{U}^c_0}{\|\boldsymbol{U}^c_0\|}$

Double specific volume limitation

- Standard limitation on $(\frac{1}{\rho})_h$ and on the Riemann invariants are performed
- Only the most limiting procedure is retained to avoid spurious oscillations

Stability

- Same stability results on the piecewise constant part W_c of the numerical solution W_h^c as for the first-order schemes
- To obtain the same stability properties on the whole piecewise polynomial solution W_h, the limitation at time t^n has to ensure that

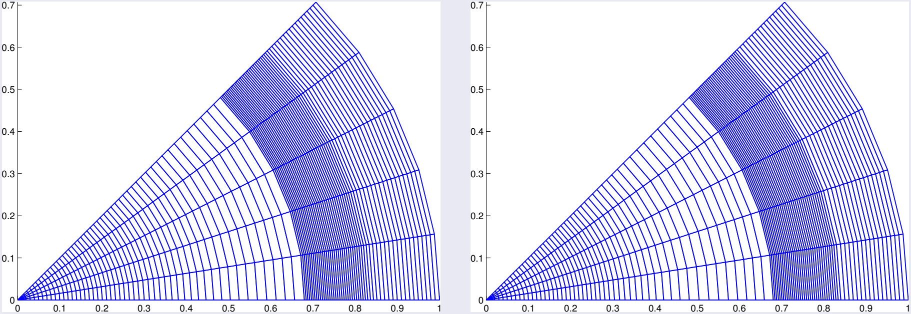
$$\forall \boldsymbol{x} \in \omega, \quad \mathrm{W}_h(\boldsymbol{x}) \in G$$

Then

- $\|(\frac{1}{\rho})_h^n\|_{L_1} = \|\frac{1}{\rho^0}\|_{L_1}$
- $\|e_h^n\|_{L_1} = \|e^0\|_{L_1}$
- $\|K_h^n\|_{L_1} < \|e_h^n\|_{L_1}$
- $\|\boldsymbol{U}_h^n\|_{L_2}^2 < m_\omega + \|e_h^n\|_{L_2}^2$

1. Cell-Centered Lagrangian schemes
2. Lagrangian and Eulerian descriptions
3. Compatible first-order positivity-preserving discretization
4. High-order positivity-preserving extension
5. Numerical results
6. Conclusion

Cylindrical Sod shock problem



(a) 1st order (b) 2nd order

Figure: Final grids on a 100x5 polar mesh, at final time $t = 1$

Cylindrical Sod shock problem

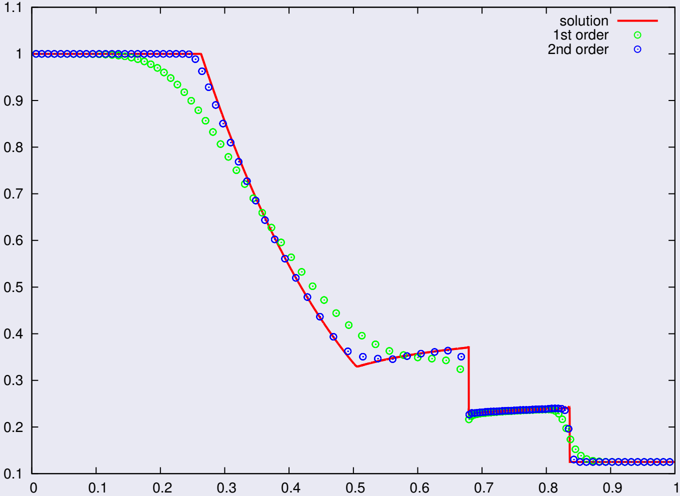
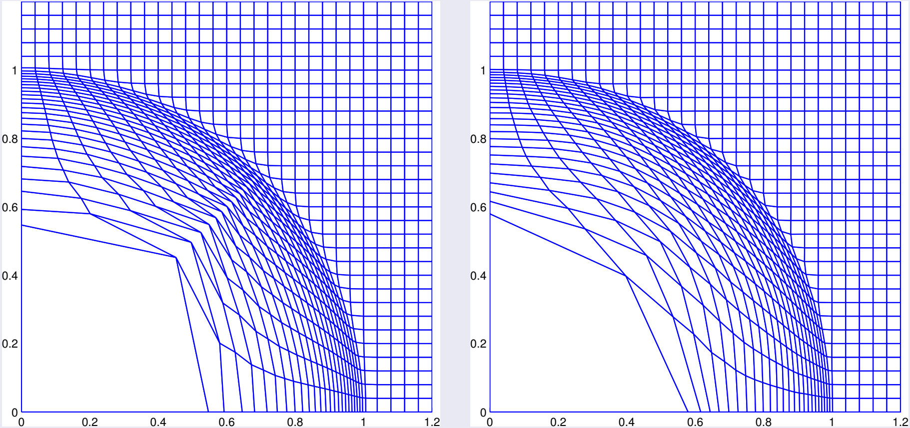


Figure: Density profile on a 100x5 polar mesh, at final time $t = 1$

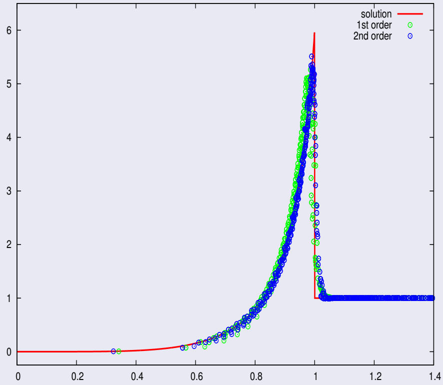
Sedov point blast problem on a Cartesian grid



(a) 1st order (b) 2nd order

Figure: Final grids on a 30x30 Cartesian mesh, at final time $t = 1$

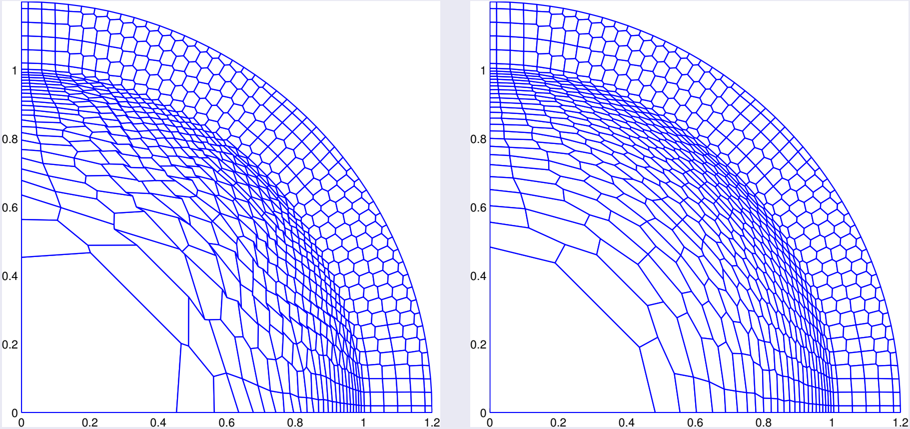
Sedov point blast problem on a Cartesian grid



(a) Density profiles (b) Pressure profiles

Figure: Density and pressure profiles on a 30x30 Cartesian mesh, at final time $t = 1$

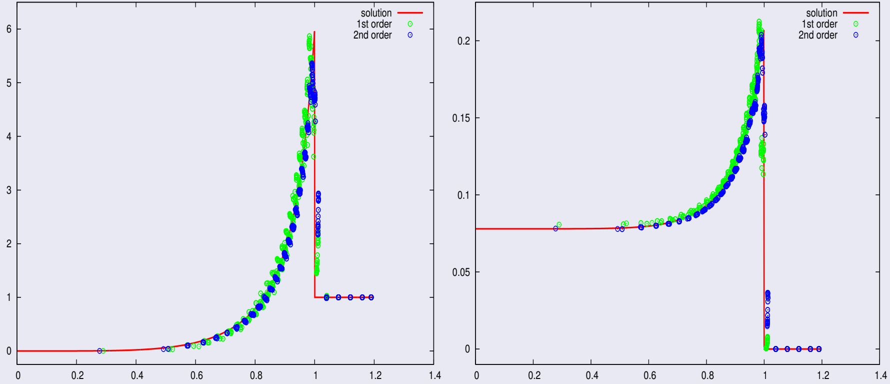
Sedov point blast problem on a polygonal grid



(a) 1st order (b) 2nd order

Figure: Final grids on mesh made of 775 polygonal cells, at final time $t = 1$

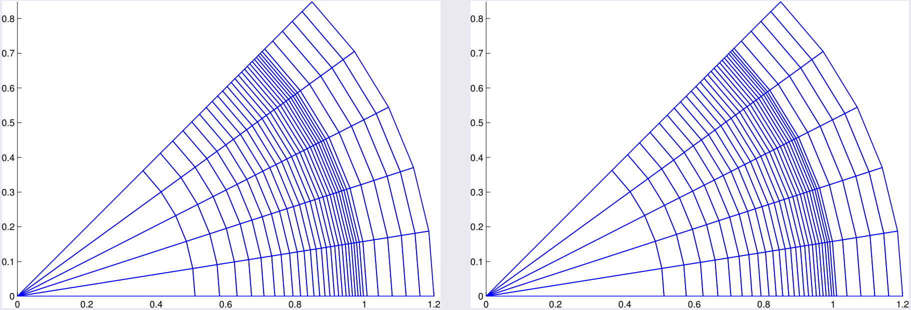
Sedov point blast problem on a polygonal grid



(a) Density profiles (b) Pressure profiles

Figure: Density and pressure profiles on mesh made of 775 polygonal cells, at final time $t = 1$

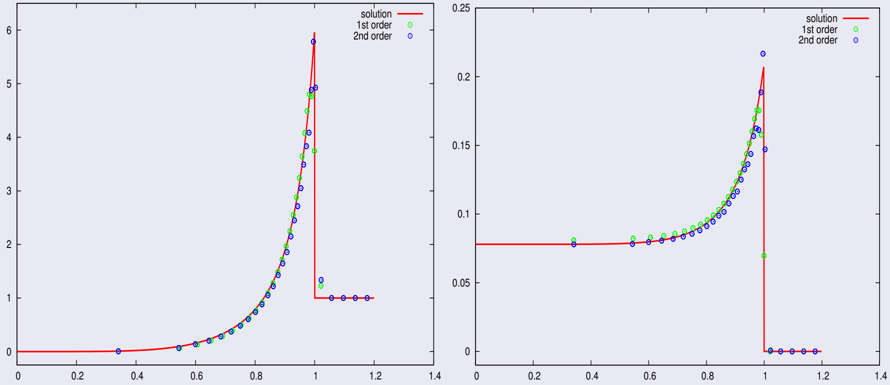
Cylindrical Sedov point blast problem



(a) 1st order (b) 2nd order

Figure: Final grids on a 30x5 polar mesh, at final time $t = 1$

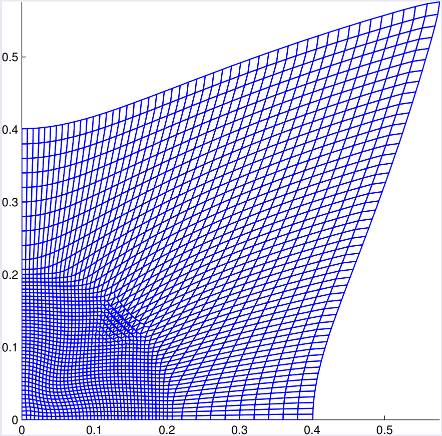
Cylindrical Sedov point blast problem



(a) Density profiles (b) Pressure profiles

Figure: Density and pressure profiles on a 30x5 polar mesh, at final time $t = 1$

Noh problem



(a) 1st order (b) 2nd order

Figure: Final grids on a Cartesian grid made of 50 × 50 cells, at final time $t = 0.6$

Noh problem

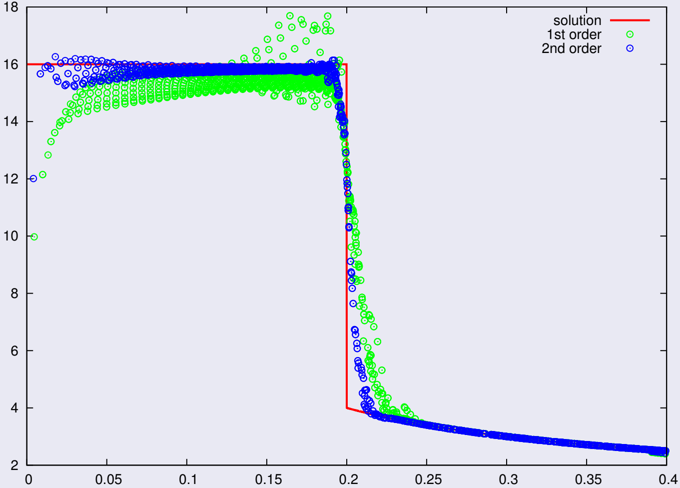
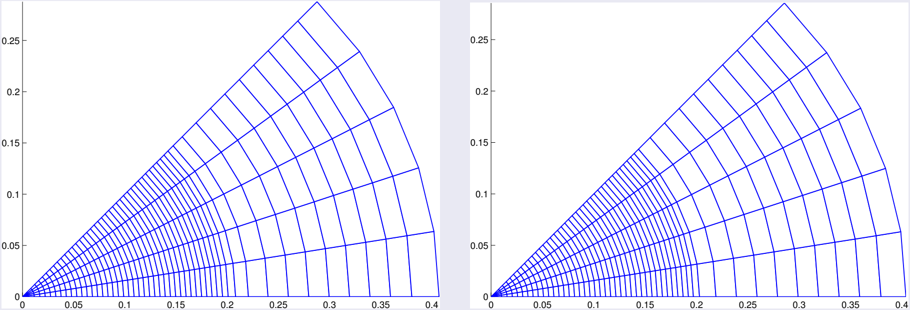


Figure: Density profile on a Cartesian grid made of 50 × 50 cells, at final time $t = 0.6$

Cylindrical Noh problem



(a) 1st order (b) 2nd order

Figure: Final grids on a 50x5 polar mesh, at final time $t = 0.6$

Cylindrical Noh problem

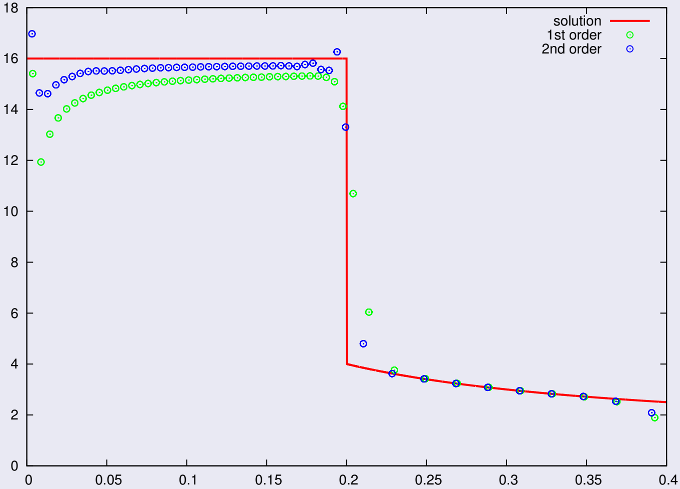
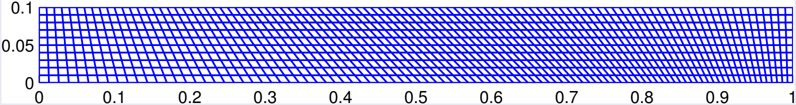
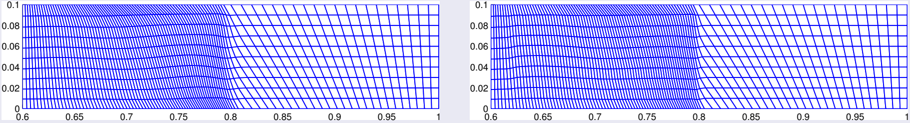


Figure: Density profile on a 50x5 polar mesh, at final time $t = 0.6$

Saltzman problem



(a) Initial mesh

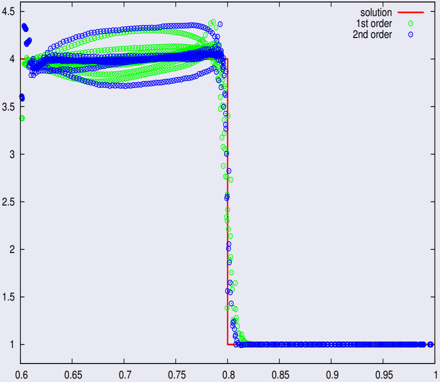


(b) 1st order

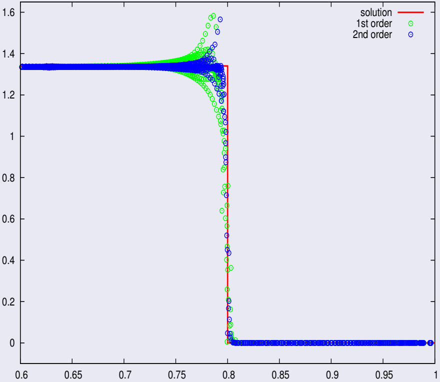
(c) 2nd order

Figure: Final grids on a 10x100 deformed Cartesian mesh, at time $t = 0.6$

Saltzman problem



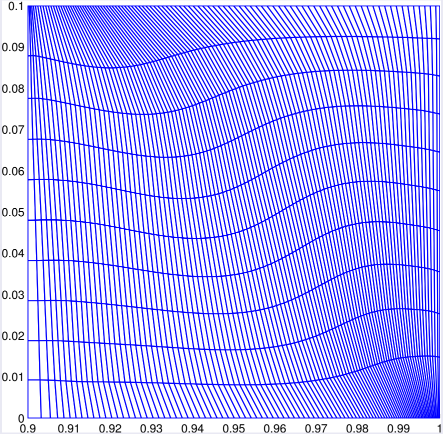
(a) Density profiles



(b) Pressure profiles

Figure: Density and pressure profiles on a 10x100 deformed Cartesian mesh, at time $t = 0.6$

Saltzman problem

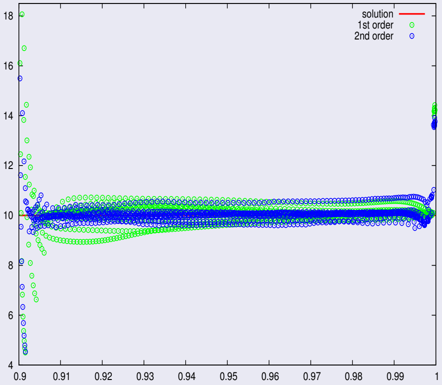


(a) 1st order

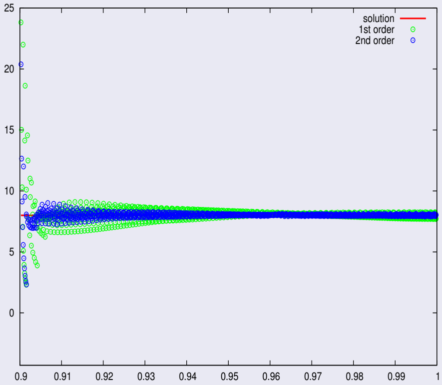
(b) 2nd order

Figure: Final grids on a 10x100 deformed Cartesian mesh, at time $t = 0.9$

Saltzman problem



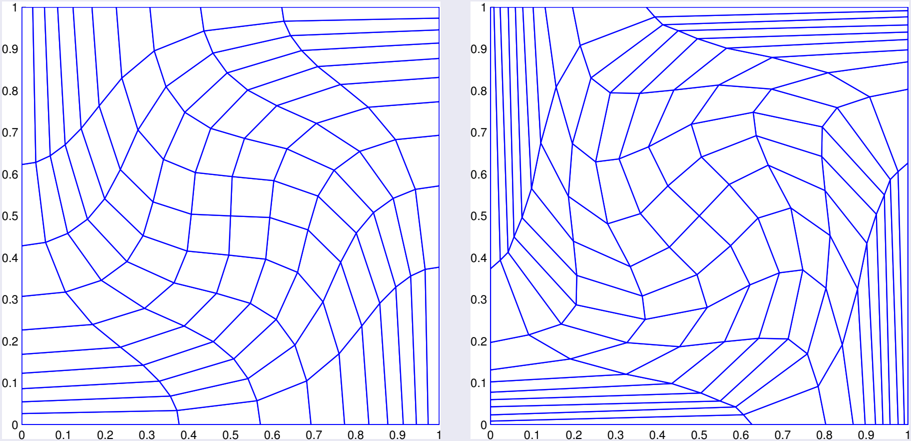
(a) Density profiles



(b) Pressure profiles

Figure: Density and pressure profiles on a 10x100 deformed Cartesian mesh, at time $t = 0.9$

Taylor-Green vortex problem



(a) 1st order (b) 2nd order

Figure: Final grids at final time $t = 0.75$, on a 10x10 Cartesian mesh

Taylor-Green vortex problem

	L_1		L_2		L_∞	
h	$E^h_{L_1}$	$q^h_{L_1}$	$E^h_{L_2}$	$q^h_{L_2}$	$E^h_{L_\infty}$	$q^h_{L_\infty}$
$\frac{1}{10}$	7.31E-2	0.97	8.90E-2	0.96	2.19E-1	0.91
$\frac{1}{20}$	3.74E-2	0.99	4.57E-2	0.98	1.17E-1	0.95
$\frac{1}{40}$	1.89E-2	0.99	2.31E-2	0.99	6.06E-2	0.97
$\frac{1}{80}$	9.50E-3	1.00	1.16E-2	1.00	3.09E-2	0.99
$\frac{1}{160}$	4.76E-3	-	5.81E-3	-	1.56E-2	-

Table: Rate of convergence computed on the velocity at time $t = 0.1$.

Taylor-Green vortex problem

	L_1		L_2		L_∞	
h	$E^h_{L_1}$	$q^h_{L_1}$	$E^h_{L_2}$	$q^h_{L_2}$	$E^h_{L_\infty}$	$q^h_{L_\infty}$
$\frac{1}{10}$	1.00E-2	2.14	1.40E-2	2.05	6.25E-2	1.58
$\frac{1}{20}$	2.27E-3	2.17	3.39E-3	2.14	2.10E-2	1.65
$\frac{1}{40}$	5.05E-4	2.14	7.66E-4	2.16	6.67E-3	1.92
$\frac{1}{80}$	1.14E-4	2.13	1.71E-4	2.16	1.76E-3	1.87
$\frac{1}{160}$	2.61E-5	-	3.83E-5	-	4.81E-4	-

Table: Rate of convergence computed on the velocity at time $t = 0.1$.

Conclusions

- Demonstration of the positivity-preserving criteria of a whole class of cell-centered Lagrangian scheme, under particular time step constraints
- Extension of the demonstration to high-order of accuracy, under particular limitation of the solution
- Demonstration of L_1 stability of the specific volume and total energy
- Control of the L_1 norm of the kinetic energy and of the L_2 norm of the velocity
- Improvement of the robustness

Perspectives

- Extension of the numerical applications to higher-order of accuracy
- Extension of the CCDG to solid dynamics such as hyperelasticity

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