Central Compact Schemes provided with Inverse Lax-Wendroff inflow boundary condition: Stability analysis

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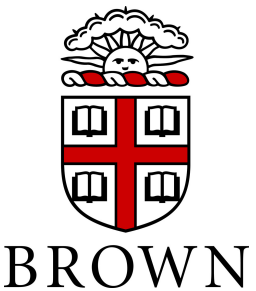
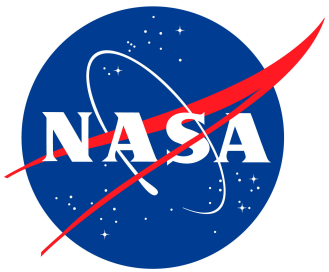




NASA project

- Wei Wang (Florida Univ.) and Mark H. Carpenter (NASA Techn. Monitor)
- Design efficient and highly accurate solvers both for direct numerical simulations and simulations of compressible flows with turbulence models

Requirements

- Good wave resolution
- High order of accuracy
- Low dissipation error

Compact schemes

- Handle non-periodic boundary conditions a lot more easily than spectral methods could
- Much smaller numerical dispersion and dissipation errors than finite difference schemes of the same order of accuracy on the same mesh

Possible issues at the boundaries

- Several ghost points near the boundary due to the wide numerical stencil
- Grids points not located on the physical boundary
- Ghost and grid points not symmetrically located with respect to the wall

Outflow boundary condition: Lagrangian extrapolation

- Ensure stability at the outflow
- Maintain the order of accuracy

Inflow boundary condition: Inverse Lax-Wendroff

- Taylor expansion of the expected order at the boundary
- Use repeatedly the partial equation (PDE) to convert the successive normal derivatives into time derivatives of the boundary condition
- Maintain the order of accuracy

S. TAN AND C.-W. SHU, *Inverse Lax-Wendroff procedure for numerical boundary conditions of conservation laws.* JCP, 2010.

Conservation laws

- We consider the solution of the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

- A semidiscrete finite difference can be represented as

$$\left(\frac{\partial u}{\partial t}\right)_i = -f(u)_i^x,$$

where $f(u)_i^x$ is the approximation of $\frac{\partial f(u)}{\partial x}$ at the grid node x_i

Lele's compact schemes (JCP 1992)

- Cell-centered compact scheme (CCCS)

$$\beta f_{i-2}^x + \alpha f_{i-1}^x + f_i^x + \alpha f_{i+1}^x + \beta f_{i+2}^x = a\frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\Delta x} + b\frac{f_{i+\frac{3}{2}} - f_{i-\frac{3}{2}}}{3\Delta x} + c\frac{f_{i+\frac{5}{2}} - f_{i-\frac{5}{2}}}{5\Delta x}$$

- Cell-node compact scheme (CNCS)

$$\beta f_{i-2}^x + \alpha f_{i-1}^x + f_i^x + \alpha f_{i+1}^x + \beta f_{i+2}^x = a\frac{f_{i+1} - f_{i-1}}{2\Delta x} + b\frac{f_{i+2} - f_{i-2}}{4\Delta x} + c\frac{f_{i+3} - f_{i-3}}{6\Delta x}$$

Lele's cell-centered compact scheme

- The resolution of the CCCS is much better than the CNCC
- Stencil contains both the grid points and half grid points
- Only the values at the cell-centers are used to calculate the derivatives at the cell-nodes

Half grid points

- Interpolation from the values at the grid points by a compact formula

$$\beta \widehat{f}_{i-\frac{5}{2}} + \alpha \widehat{f}_{i-\frac{3}{2}} + \widehat{f}_{i+\frac{1}{2}} + \alpha \widehat{f}_{i+\frac{3}{2}} + \beta \widehat{f}_{i+\frac{5}{2}} = a\frac{f_{i+1} + f_i}{2} + b\frac{f_{i+2} + f_{i-1}}{2} + c\frac{f_{i+3} + f_{i-2}}{2}$$

- Introduce transfer errors
- Significantly reduces the resolution for high wave numbers

S.K. LELE, *Compact finite difference schemes with spectral-like resolution.* JCP, 1992.

Main idea

- If both cell-node and cell-center values are used to compute the derivatives, one could get higher order of accuracy and better resolution

New class of central compact schemes

- Central Compact Schemes (CCS)

$$\beta f'_{i-2} + \alpha f'_{i-1} + f'_i + \alpha f'_{i+1} + \beta f'_{i+2}$$
$$=$$
$$a\frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\Delta x} + b\frac{f_{i+1} - f_{i-1}}{2\Delta x} + c\frac{f_{i+\frac{3}{2}} - f_{i-\frac{3}{2}}}{3\Delta x} + d\frac{f_{i+2} - f_{i-2}}{4\Delta x} + e\frac{f_{i+\frac{5}{2}} - f_{i-\frac{5}{2}}}{5\Delta x}$$

- Both CCCS and CNCS of Lele are special cases of these CCS


X. Liu, S. Zhang, H. Zhang and C.-W. Shu, *A new class of central compact schemes with spectral-like resolution I: Linear schemes.* JCP, 2013.

Half grid points

- Stored as independent computational variables
- Computed using the same scheme, shifting the indices by $\frac{1}{2}$

$$\beta f^x_{i-\frac{3}{2}} + \alpha f^x_{i-\frac{1}{2}} + f^x_{i+\frac{1}{2}} + \alpha f^x_{i+\frac{3}{2}} + \beta f^x_{i+\frac{5}{2}}$$
$$=$$
$$a\frac{f_{i+1} - f_i}{\Delta x} + b\frac{f_{i+\frac{3}{2}} - f_{i-\frac{1}{2}}}{4\Delta x} + c\frac{f_{i+2} - f_{i-1}}{3\Delta x} + d\frac{f_{i+\frac{5}{2}} - f_{i-\frac{3}{2}}}{4\Delta x} + e\frac{f_{i+3} - f_{i-2}}{5\Delta x}$$

Outcome

- Gain in accuracy and resolution
- Double memory requirement in 1D
- **But same cost as compact interpolation**

Reformulation on a twice more refined mesh

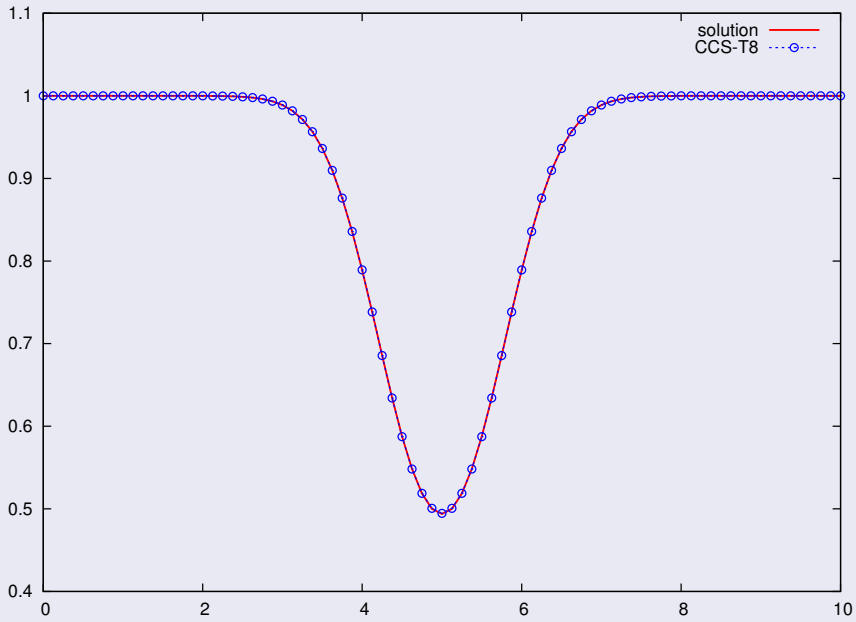
- CCS rewrite as cell-node compact schemes as

$$\beta f^x_{i-4} + \alpha f^x_{i-2} + f^x_i + \alpha f^x_{i+2} + \beta f^x_{i+4}$$
$$=$$
$$a\frac{f_{i+1} - f_{i-1}}{2\Delta x} + b\frac{f_{i+2} - f_{i-2}}{4\Delta x} + c\frac{f_{i+3} - f_{i-3}}{6\Delta x} + d\frac{f_{i+4} - f_{i-4}}{8\Delta x} + e\frac{f_{i+5} - f_{i-5}}{10\Delta x}$$

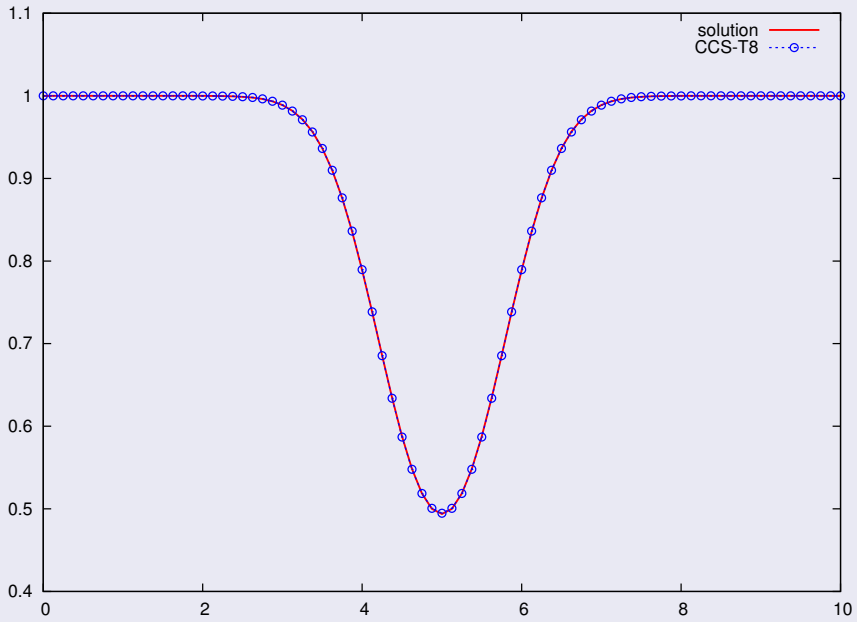
Time discretization

- Third-order TVD Runge-Kutta

Two dimensional Euler equations



(a) Final time $t = 50$



(b) Final time $t = 200$

Figure : The distribution of the density along $x = 5$ for the two dimensional advection of an isentropic vortex on a 80×80 Cartesian grid, with CCS-T8.

Initial Boundary Value Problem (IBVP)

- We consider the following initial boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0, & x \in [x_A, x_B],\ t \geq 0, \\ u(x_A, t) = g(t), & t \geq 0, \\ u(x, 0) = u_0(x), & x \in [x_A, x_B] \end{cases}$$

- We assume $f'(u(x_A, t)) > 0$ and $f'(u(x_B, t)) > 0$, where $f'(u) = \frac{\mathrm{d}f(u)}{\mathrm{d}u}$

Cartesian grid

- Uniform mesh $\{x_j\}_{j=0,\ldots,n}$ such as

$$x_0 - C_A \Delta x = x_A \leq x_0 < x_1 < \cdots < x_n \leq x_B = x_n + C_B \Delta x$$

  where $C_A \in [0, 1]$ and $C_B \in [0, 1]$
- The grid points x_0 and x_n are not necessarily located on the boundaries x_A and x_B

Outflow: f_{n+p}

- f_{n+p} have to be defined, for $p = 1, \ldots, 5$
- s^{th} order extrapolation procedure is used

$$f_{n+p} = f(u_{n+p}), \qquad u_{n+p} = \sum_{j=1}^{s} u_{n-s+j} \prod_{\substack{l=1 \\ l \neq j}}^{s} \left(\frac{p+s-l}{j-l} \right)$$

M. GOLDBERG, *On a boundary extrapolation theorem by Kreiss.* 1977.

Outflow: f^x_{n+p}

- f^x_{n+p} have to be defined, for $p = 1, \ldots, 4$
- Extension of the extrapolation procedure to the derivative

$$f^x_{n+p} = f'(u_{n+p}) \left. \frac{\partial u}{\partial x} \right|_{n+p},$$

$$\left. \frac{\partial u}{\partial x} \right|_{n+p} = \frac{1}{\Delta x} \sum_{j=1}^{s} u_{n-s+j} \prod_{\substack{l=1 \\ l \neq j}}^{s} \left(\frac{p+s-l}{j-l} \right) \sum_{\substack{q=1 \\ q \neq j}}^{s} \left(\frac{1}{p+s-q} \right)$$

Inflow: f_{-p}

- f_{-p} have to be defined, for $p = 1, \ldots, 5$
- Inverse Lax-Wendroff (ILW) procedure is used
  - Taylor expansion at the boundary x_A

$$f_{-p} = f(u_{-p}), \qquad u_{-p} = \sum_{k=0}^{s-1} \frac{(x_{-p} - x_A)^k}{k!} \, u^{*\,(k)},$$

where $u^{*\,(k)}$ are the $(s-k)^{\text{th}}$ order approximation of $\frac{\partial^k u}{\partial x^k}|_{x_A}$

  - Repetitive use of the PDE to convert spatial derivatives to time derivatives

$$u^{*\,(0)} = u(x_A, t) = g(t),$$
$$u^{*\,(1)} = \frac{\partial u}{\partial x}|_{x_A} = -\frac{g'(t)}{f'(g(t))},$$
$$u^{*\,(2)} = \frac{\partial^2 u}{\partial x^2}|_{x_A} = \frac{f'(g(t))g''(t) - 2f''(g(t))g'(t)^2}{f'(g(t))^3}.$$

S. TAN AND C.-W. SHU, *Inverse Lax-Wendroff procedure for numerical boundary conditions of conservation laws.* JCP, 2010.

Inflow: f^x_{-p}

- f^x_{-p} have to be defined, for $p = 1, \ldots, 4$
- Extension of the ILW procedure to the derivative

$$f^x_{-p} = f'(u_{-p}) \, \frac{\partial u}{\partial x}|_{-p},$$

$$\frac{\partial u}{\partial x}|_{-p} = \sum_{k=0}^{s-2} \frac{(x_{-p} - x_A)^k}{k!} \, u^{*\,(k+1)}$$

where the $u^{*\,(k)}$ have already been computed in the evaluation of u_{-p}

Outcome

- Very heavy algebra for very high order of approximation, or for fully nonlinear systems of equations
- Simplified version of the ILW procedure with only to two leading terms

S. TAN, C. WANG, C.-W. SHU AND J. NING, *Efficient implementation of high order inverse Lax-Wendroff boundary treatment for conservation laws.* JCP, 2012.

Extension of the Simplified Inverse Lax-Wendroff (SILW)

- The first k_s moments are computed by the ILW procedure, and the following ones through the use of an extrapolation

$$u_{-p} = \sum_{k=0}^{k_s-1} \frac{(-p+C_A)^k}{k!} \Delta x^k \; u_{ILW}^{*\,(k)} + \sum_{k=k_s}^{s-1} \frac{(-p+C_A)^k}{k!} \Delta x^k \; u_{EXT}^{*\,(k)},$$

where the successive moments $u_{EXT}^{*\,(k)}$ write

$$u_{EXT}^{*\,(k)} = \sum_{j=1}^{s} \frac{u_{j-1}}{\Delta x^k} \prod_{\substack{l=1 \\ l\neq j}}^{s} \left(\frac{1-C_A-l}{j-l} \right) \sum_{\substack{q_1=1 \\ q_1\neq j}}^{s} \left(\frac{1}{1-C_A-q_1} \right) \cdots \sum_{\substack{q_k=1 \\ q_k\neq j \\ q_k\neq q_1,\ldots,q_{k-1}}}^{s} \left(\frac{1}{1-C_A-q_k} \right)$$

- Same procedure on the derivative

$$\frac{\partial u}{\partial x}|_{-p} = \sum_{k=0}^{k_s-2} \frac{(-p+C_A)^k}{k!} \Delta x^k \; u_{ILW}^{*\,(k+1)} + \sum_{k=k_s-1}^{s-2} \frac{(-p+C_A)^k}{k!} \Delta x^k \; u_{EXT}^{*\,(k+1)}$$

Theorem 1

G-K-S theory asserts that to show **stability** for the finite-domain problem, it is sufficient to show that **the inner scheme is Cauchy stable on** $(-\infty, +\infty)$, and that **each of the two quarter-plane problems is stable with the use of normal mode analysis**. Thus, the stability of the finite-domain problem is broken into the summation of three simpler problems

Theorem 2

For each quarter-plane problem that arise from Theorem 1, a **necessary and sufficient** condition for stability of the IBVP is that **no eigensolution exists**. This theorem is true for either the fully discrete case or the semidiscrete case.

References

B. GUSTAFSSON, H.-O. KREISS AND A. SUNDSTRÖM, *Stability theory of difference approximations for mixed initial boundary value problem. II.* Math. of Comp., 1972.

J.C. STRIKWERDA, *Initial boundary value problems for the method of lines.* JCP, 1980.

Quarter-plane problem

- We consider the following quarter-plane problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + A \dfrac{\partial u}{\partial x} = 0, & x \geq 0,\ t \geq 0, \\ u(0,t) = g(t), & t \geq 0, \quad \textbf{if } \boldsymbol{A > 0}, \\ u(x,0) = u_0(x), & x \geq 0, \\ \|u(.,t)\| < \infty, \end{cases}$$

  where $\|u(.,t)\| = \int_0^\infty |u(x,t)|^2 \mathrm{d}x$

- Uniform grid $0 \leq x_0 = C_0\,\Delta x < x_1 < \ldots, \quad C_0 \in [0,1[$
- The use of compact scheme yields the semidiscrete inner scheme

$$\mathcal{P}\frac{\mathrm{d}\,u_j}{\mathrm{d}t} = -\frac{A}{\Delta x}\mathcal{Q}u_j, \quad \text{for } j = r, r+1, \ldots$$

where $\mathcal{P} = \sum_{i=-r_L}^{p_L} \alpha_i E^i$, $\mathcal{Q} = \sum_{i=-r_R}^{p_R} a_i E^i$, $E^i u_j = u_{j+i}$ and $r = \max(r_L, r_R)$

Quarter-plane problem

- Quarter-plane problem discretization

$$\begin{cases} \mathcal{P}\dfrac{\mathrm{d}\,u_j}{\mathrm{d}t} = -\dfrac{A}{\Delta x}\mathcal{Q}u_j, & t \geq 0, \quad j = r, r+1, \ldots \\ \mathcal{D}_j\dfrac{\mathrm{d}\,u_j}{\mathrm{d}t} = -\dfrac{A}{\Delta x}\mathcal{B}_j u_j + \widetilde{g}_j(t), & t \geq 0, \quad j = 0, 1, \ldots, r-1 \\ u_j(0) = u_0(x_j), & j = 0, 1, \ldots \\ \sum_{j=0}^{\infty} |u_j(t)|^2 \Delta x < \infty, & t \geq 0 \end{cases}$$

Definition

An eigensolution is the nontrivial function $v(x,s) = e^{st}\phi(x)$, which satisfies

a) $s\,\Delta x\,\mathcal{P}v_j + A\,\mathcal{Q}v_j = 0, \quad j = r, r+1, \ldots$

b) $Re(s) \geq 0$

c) For $Re(s) > 0$, $v(x,s)$ is bounded as $x \to \infty$

d) For $Re(s) = 0$, $v(x,s) = \lim_{\varepsilon \to 0^+} v(x, s+\varepsilon)$, where $v(x, s+\varepsilon)$ satisfies a) and c) with respect with $s + \varepsilon$

e) $s\,\Delta x\,\mathcal{D}_j v_j + A\,\mathcal{B}_j v_j = \widetilde{g}_j(t), \quad j = 0, 1, \ldots, r-1$

Example: CCS-T6

- Inner scheme

$$-\frac{u^x_{j-2}}{12} + u^x_j - \frac{u^x_{j+2}}{12} = \frac{16}{9}\frac{u_{j+1} - u_{j-1}}{2\Delta x} - \frac{17}{18}\frac{u_{j+2} - u_{j-2}}{4\Delta x}$$

- u_{-p} and u^x_{-p}, for $p = 1, 2$, are evaluated by extrapolation in the outflow case, and ILW or SILW in the inflow case

Normal mode analysis: $u_j(t) = e^{s\,t}\,\phi_j$ where $\phi_j = C\,K^j$

- Characteristic equation, with $\widetilde{s} = s\,\frac{\Delta x}{|A|}$

$$\widetilde{s}\left(K^2 - \frac{1}{12}\left(K^4 + 1\right)\right) + sgn(A)\left(\frac{16}{18}\left(K^3 - K\right) - \frac{17}{72}\left(K^4 - 1\right)\right) = 0$$

- Only two roots of the resolvent equation yield $|K| \leq 1$
- Thus, the general solution has the form

$$\phi_j = C_1\,K_1^j + C_2\,K_2^j,$$

where K_1, K_2 the two roots with $|K| < 1$ and C_1, C_2 two constants

Boundary conditions

- Substituting the general solution into the two boundary conditions for $j = 0$ and $j = 1$ yield a 2×2 system of equations
- For example, in the outflow case ($A = -1$) with extrapolation boundary

$$\widetilde{s}\,(72\phi_0 - 6\phi_2) + \frac{1647}{10}\phi_0 - 363\phi_1 + 358\phi_2 - 234\phi_3 + \frac{177}{2}\phi_4 - \frac{71}{5}\phi_5 = 0$$

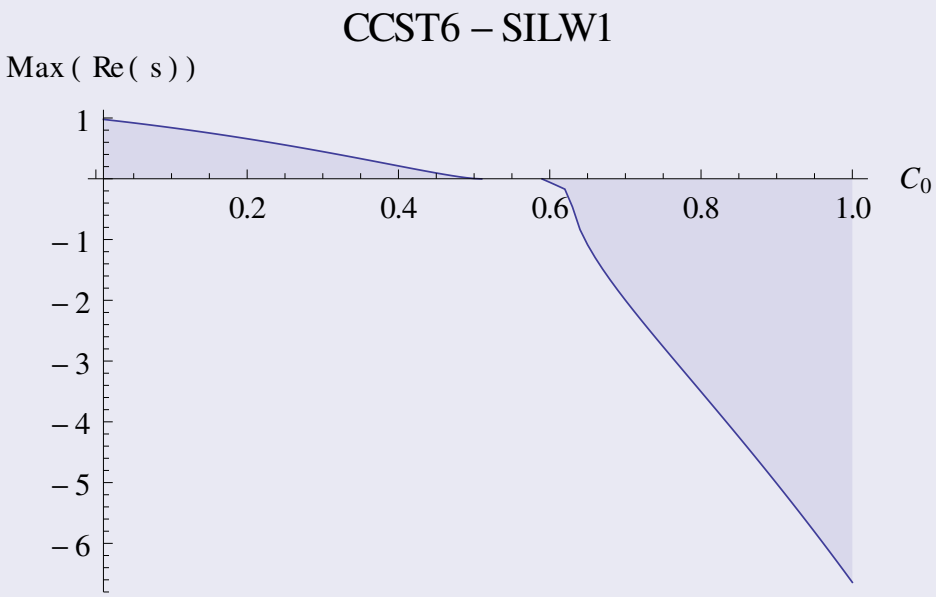
$$\widetilde{s}\,(72\phi_1 - 6\phi_3) - \frac{71}{5}\phi_0 + \frac{159}{2}\phi_1 - 150\phi_2 + 74\phi_3 - 21\phi_4 + \frac{33}{10}\phi_5 = 0$$

- This system has only a trivial solution unless its determinant is null

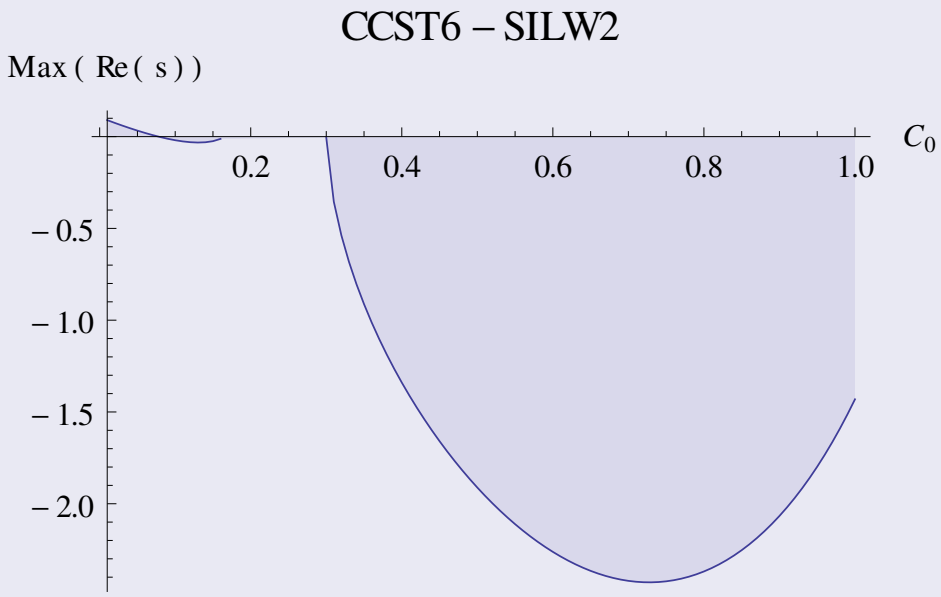
Outcome

- The **extrapolation outflow** boundary condition maintain the stability for any CCS and any value of C_0 (no eigensolution)
- The **ILW inflow** boundary condition maintain the stability for any CCS and any value of C_0 (no eigensolution)
- The stability of the CCS with the **SILW inflow** boundary condition **depends on** the value of C_0 and on the **number of leading terms** k_s

CCS-T6 provided with SILW inflow boundary



(a) One leading term



(b) Two leading terms

Figure : Maximum of the real part of the eigenvalues as function of C_0 for the CCS-T6 scheme with the SILW boundary condition with one and two leading terms.

CCS-T6 provided with SILW inflow boundary

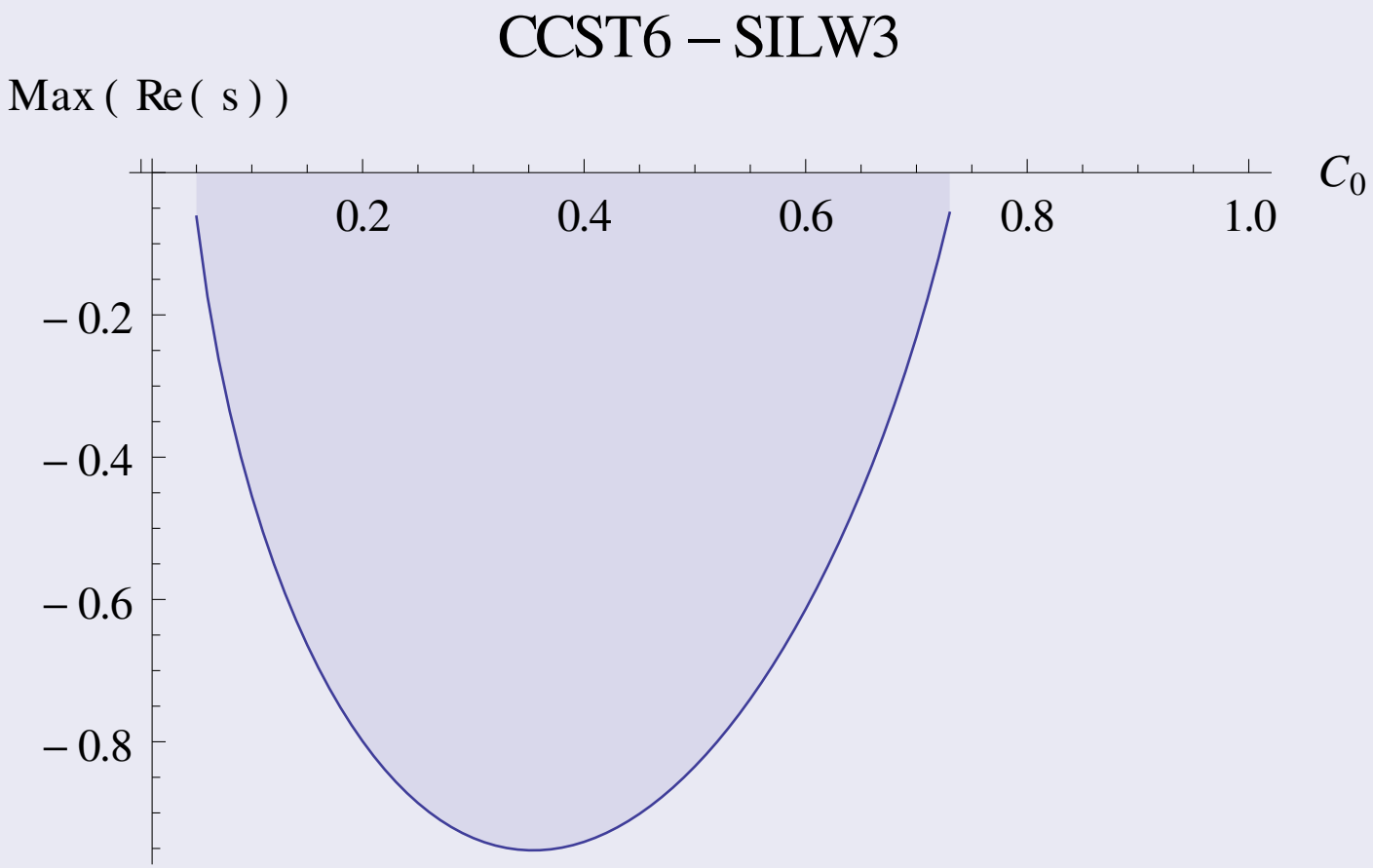


Figure : Maximum of the real part of the eigenvalues as function of C_0 for the CCS-T6 scheme with the SILW boundary condition with three leading terms.

Time discretization

- We use the third-order TVD Runge-Kutta method
- Let us consider the general system

$$\frac{\mathrm{d}\,u}{\mathrm{d}t} = F(t,u)$$

- We derive the eigenvalue problem setting $F(t,u) = s\,u$

$$u^{n+1} = (1 + \mu + \frac{\mu^2}{2} + \frac{\mu^3}{6})\,u^n,$$

where $u^n = u(x,t^n)$ and $\mu = s\,\Delta t = \widetilde{s}\,\frac{|A|\,\Delta t}{\Delta x}$

- This is nothing but a Taylor expansion of the exponential e^{μ}
- Assuming a solution of the form $u^n = z^n\,u^0$, where z is a complex number, the stability domain of the considered time discretization writes

$$|z(\mu)| \leq 1, \quad \text{where} \quad z(\mu) = 1 + \mu + \frac{\mu^2}{2} + \frac{\mu^3}{6}$$

Normal mode analysis

- Semidiscrete case: $u_j(t^{n+1}) = e^{s\,\Delta t}\, u_j(t^n) = e^{\widetilde{s}\,\frac{|A|\,\Delta t}{\Delta x}}\, u_j(t^n)$

  $Re(s) \leq 0$ and s is not a generalized eigenvalue $\Longrightarrow$ Stability
- Fully discrete case: $u_j^{n+1} = z(s\,\Delta t)\, u_j^n = z(\widetilde{s}\,\frac{|A|\,\Delta t}{\Delta x})\, u_j^n$

  $|z| \leq 1$ and z is not a generalized eigenvalue $\Longrightarrow$ Stability
- We introduce the CFL condition: $CFL = \frac{|A|\,\Delta t}{\Delta x}$
- We substitute in the time discretization resolvent equation the eigenvalues $\widetilde{s}$ we have found in the semidiscrete G-K-S analysis

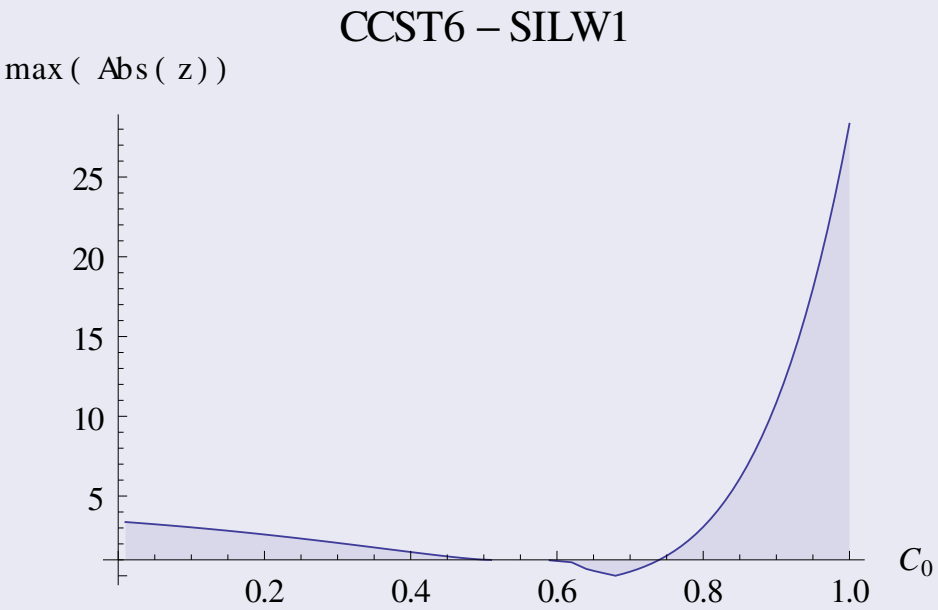
Procedure

- We start the stability analysis with the same CFL condition than for the periodic boundary case
- If the fully discrete scheme is not stable under this CFL ($\exists\, z,\ |z| \geq 1$), we use a decreasing sequence of CFL condition,re-performing at each step to stability analysis

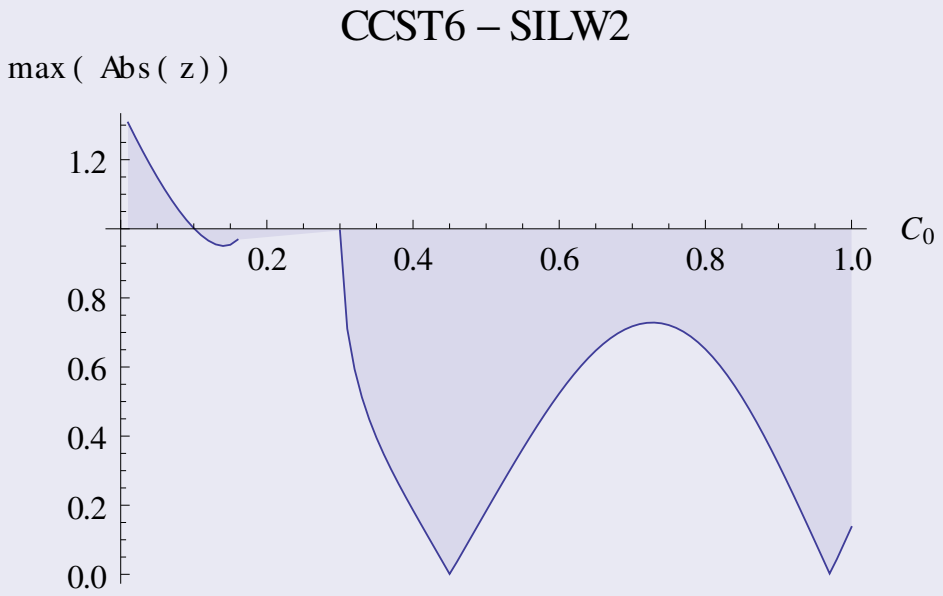
Outcome

- The **extrapolation outflow** boundary condition maintain the stability for any RK3-CCS and any value of C_0 (no eigensolution), **under the same CFL** than for the periodic boundary case
- The **ILW inflow** boundary condition maintain the stability for any RK3-CCS and any value of C_0 (no eigensolution), **under the same CFL** than for the periodic boundary case
- The stability of the CCS with the **SILW inflow** boundary condition **depends on** the value of C_0 and of the **number of leading terms** k_s. The fully discrete scheme would be **stable under the same CFL** than for the periodic boundary case **or not stable for any CFL**, depending on the number of leading terms

RK3-CCST6 provided with SILW inflow boundary



(a) One leading term



(b) Two leading terms

Figure : Maximum of the absolute value of the eigenvalues as function of C_0 for the RK3-CCST6 scheme with the SILW boundary condition with one and two leading terms.

RK3-CCST6 provided with SILW inflow boundary

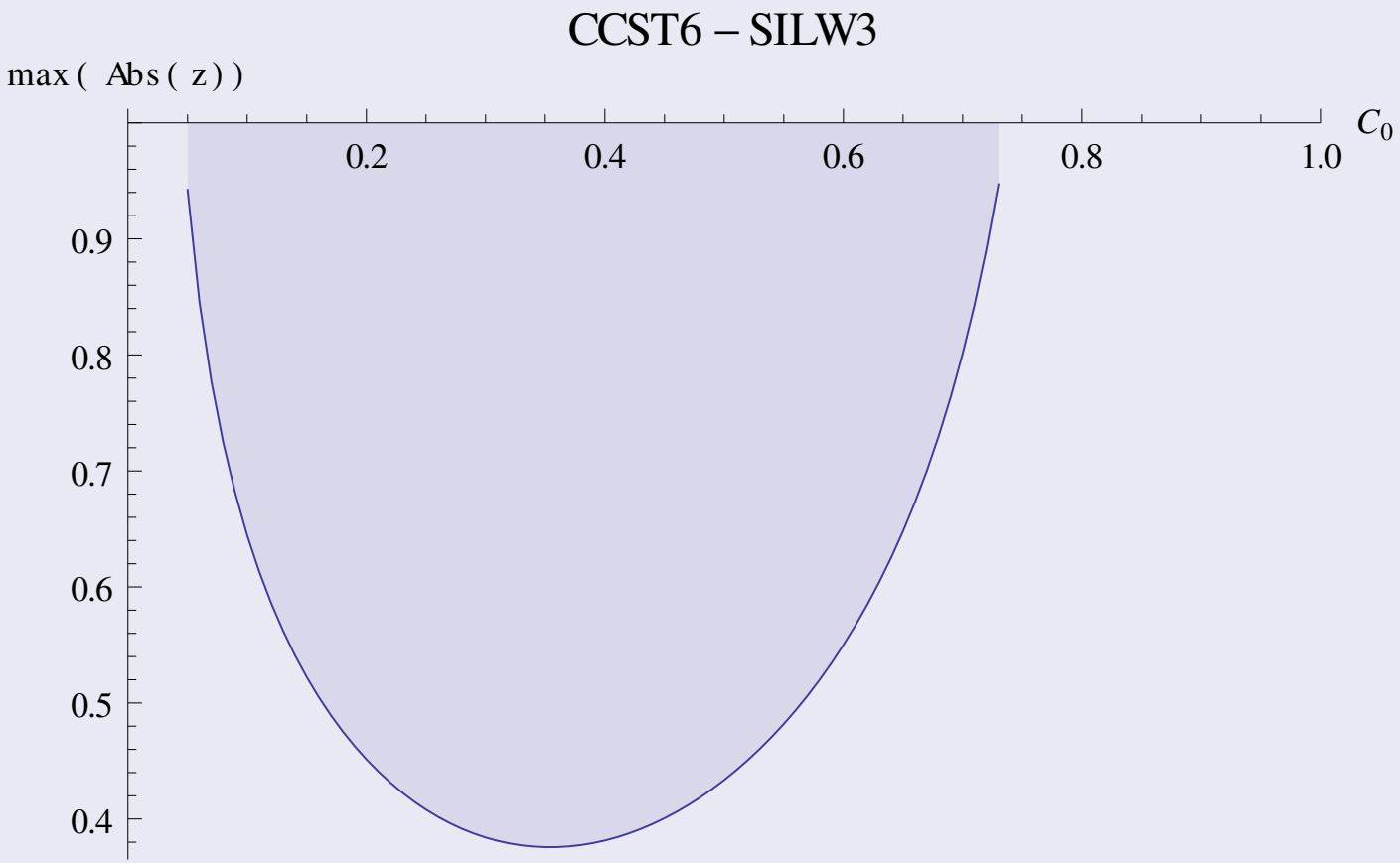


Figure : Maximum of the absolute value of the eigenvalues as function of C_0 for the RK3-CCST6 scheme with the SILW boundary condition with three leading terms.

G-K-S theory advantages

- Analytical analysis of the stability
- The stability of the finite-domain problem is broken into the summation of three simpler problems
- Analysis independent of the mesh resolution

G-K-S theory disadvantages

- Complex theory
- **Very heavy algebra** (not practical for very high-order of accuracy)

Initial Boundary Value Problem

- We consider the following initial boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + A\dfrac{\partial u}{\partial x} = 0, & x \in [x_A, x_B],\ t \geq 0, \\ u(x_A, t) = g(t), & t \geq 0, \\ u(x, 0) = u_0(x), & x \in [x_A, x_B], \end{cases}$$

- We assume $A > 0$

Cartesian grid

- Uniform mesh $\{x_j\}_{j=0,\dots,n}$ such as

$$x_0 - C_A\Delta x = x_A \leq x_0 < x_1 < \cdots < x_n \leq x_B = x_n + C_B\Delta x$$

  where $C_A \in [0, 1]$ and $C_B \in [0, 1]$
- The grid points x_0 and x_n are not necessarily located on the boundaries x_A and x_B

Discretization

- Central compact scheme is used at the inner points
- The ghost points located at the outflow boundary condition are evaluated by an extrapolation procedure
- The ghost points located at the inflow boundary condition are evaluated either by the ILW procedure or its simplified version SILW
- Finally, the semidiscrete scheme yields a linear system of equations expressed in a matrix-vector form as

$$\mathrm{P}\frac{\mathrm{d}\,\boldsymbol{U}}{\mathrm{d}t} = -\frac{A}{\Delta x}\mathrm{Q}\boldsymbol{U},$$

where P is invertible and $\boldsymbol{U} = (u_0, u_1, \ldots, u_n)^{\mathrm{t}}$

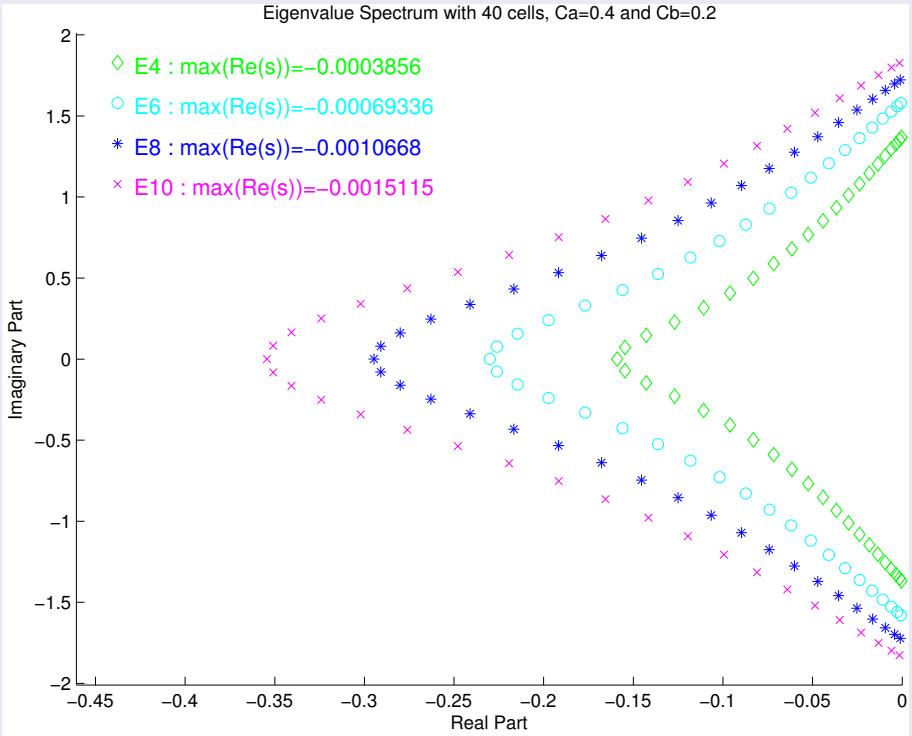
Normal mode analysis

- Assuming a solution of the form $u(x,t) = e^{s\,t}\,u^0(x)$, the semidiscrete scheme yields

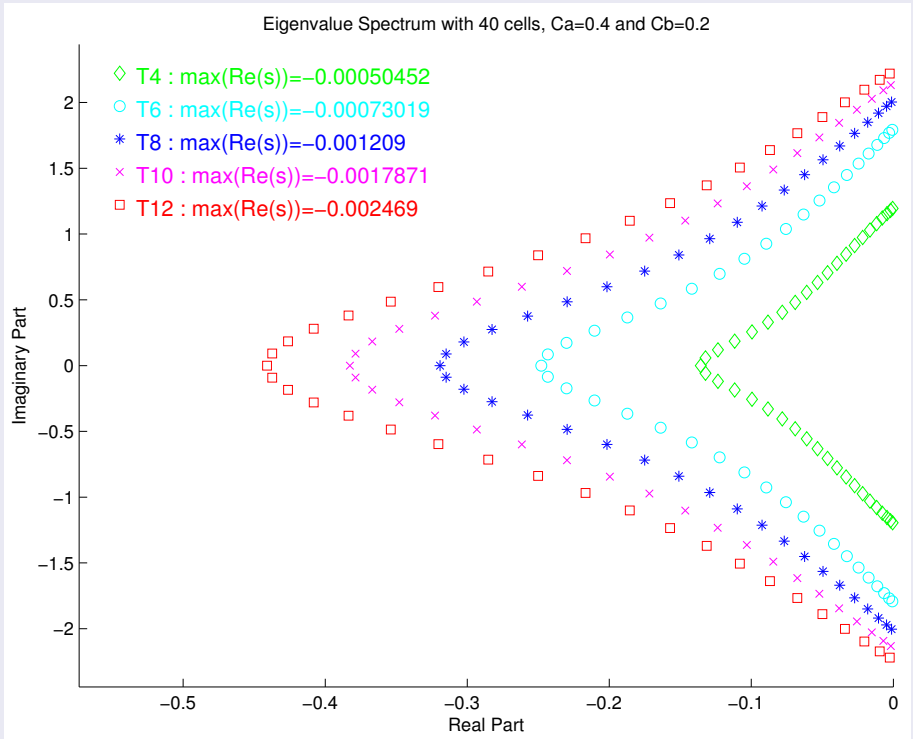
$$\widetilde{s}\,\mathrm{P}\boldsymbol{U} = -sgn(A)\,\mathrm{Q}\boldsymbol{U},$$

  where $\widetilde{s} = s\,\frac{\Delta x}{|A|}$ being the considered eigenvalue
- Thus, we compute the eigenvalues of matrix $-sgn(A)\,\mathrm{P}^{-1}\mathrm{Q}$
- As previously, the semidiscrete scheme provided with the considered boundary conditions, on the studied mesh, is stable if the whole eigenvalue spectrum lies in the left-hand plane ($Re(\widetilde{s}) \leq 0$)

CCS provided with extrapolation-ILW boundary conditions



(a) CCS-E



(b) CCS-T

Figure : The eigenvalue spectrum of the semi-discrete central compact schemes, closed with an Inverse Lax-Wendroff procedure for the inflow boundary, and extrapolation for the outflow boundary, with 40 cells and $C_A = 0.4$ and $C_B = 0.2$.

CCS-P provided with extrapolation-ILW boundary conditions

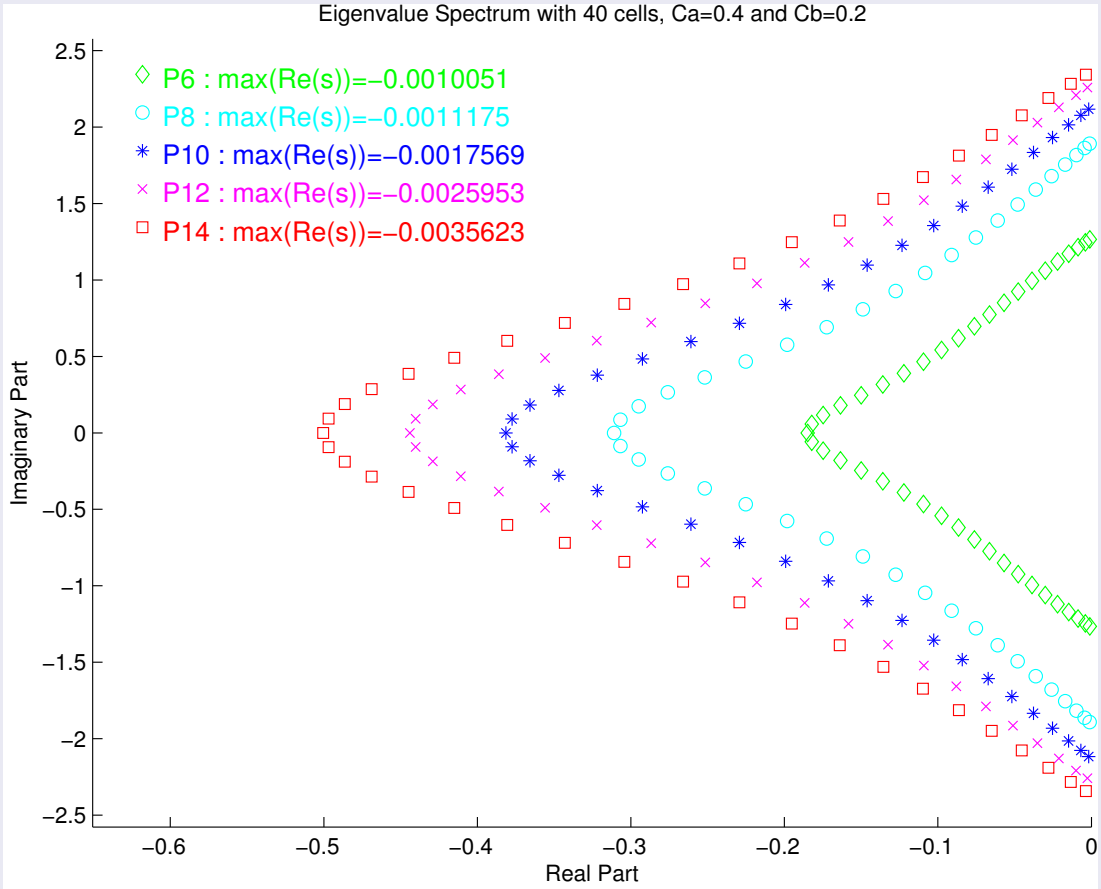


Figure : The eigenvalue spectrum of the semi-discrete CCS-P schemes, closed with an Inverse Lax-Wendroff procedure for the inflow boundary, and extrapolation for the outflow boundary, with 40 cells and $C_A = 0.4$ and $C_B = 0.2$.

CCS-T6 provided with extrapolation-SILW1 boundary conditions

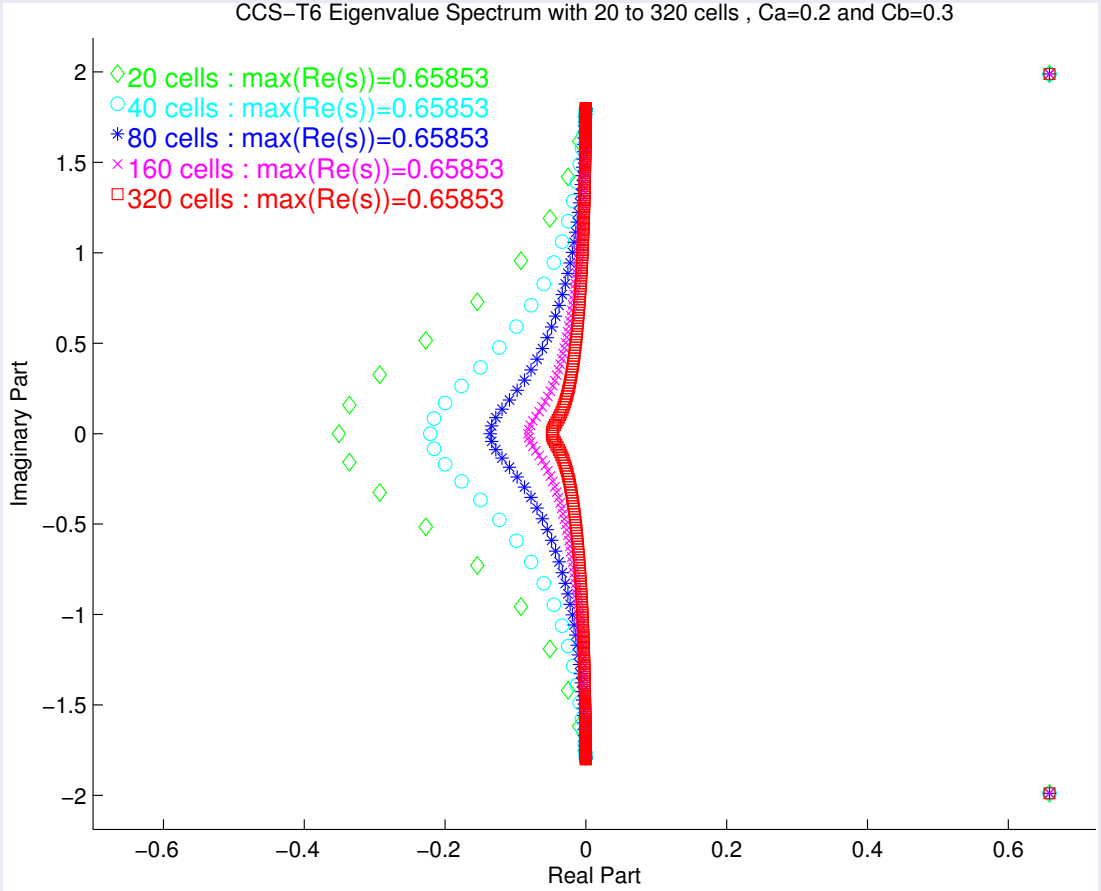
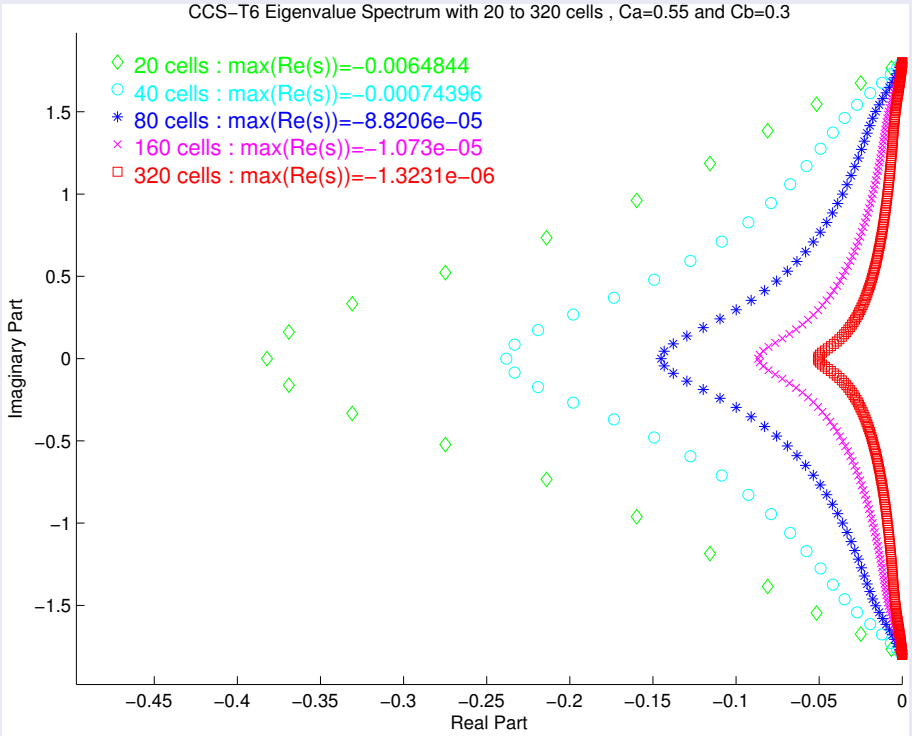
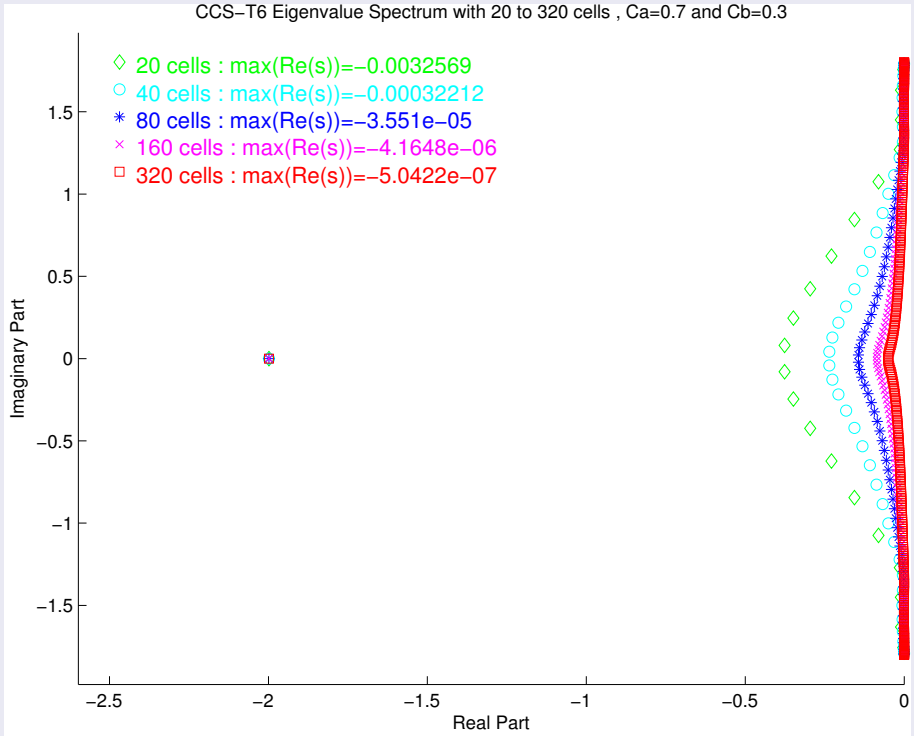


Figure : The eigenvalue spectrum of the CCS-T6, provided with SILW procedure with one term for the inflow boundary, and extrapolation for the outflow boundary, with $C_A = 0.2$.

CCS-T6 provided with extrapolation-SILW1 boundary conditions



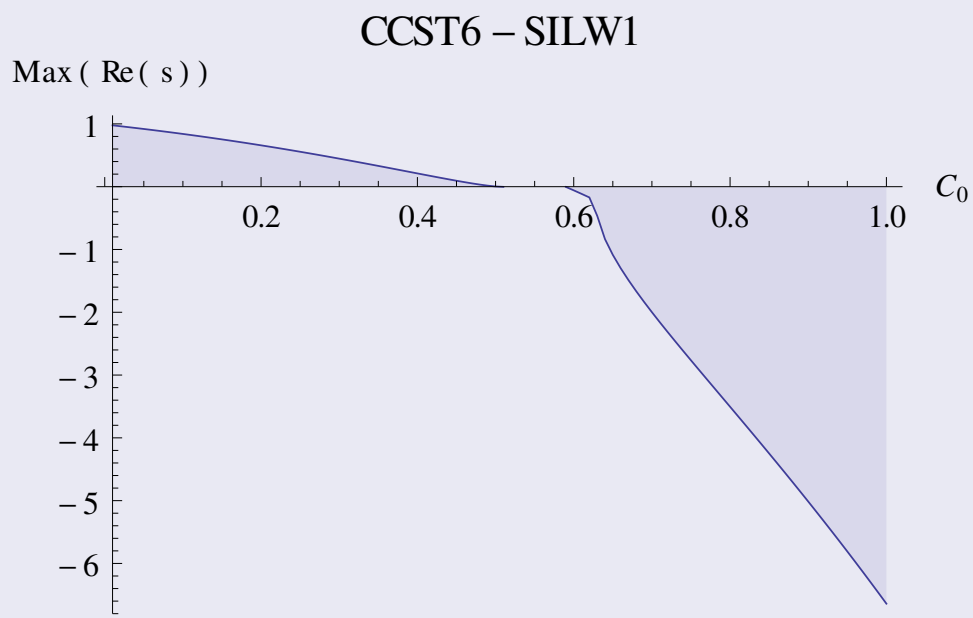
(a) $C_A = 0.55$



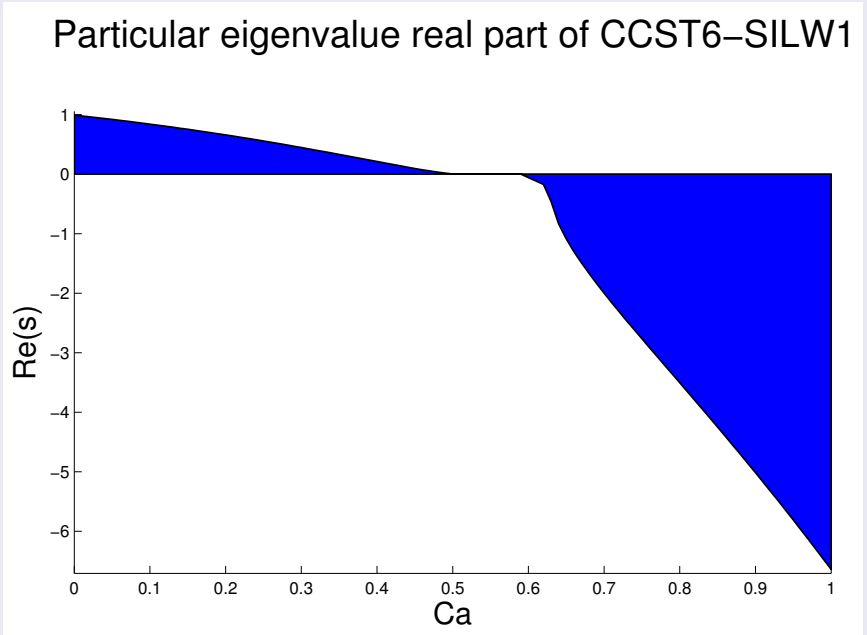
(b) $C_A = 0.7$

Figure : The eigenvalue spectrum of the CCS-T6, provided with SILW procedure with one term for the inflow boundary, and extrapolation for the outflow boundary.

CCS-T6 provided with extrapolation-SILW1 boundary conditions



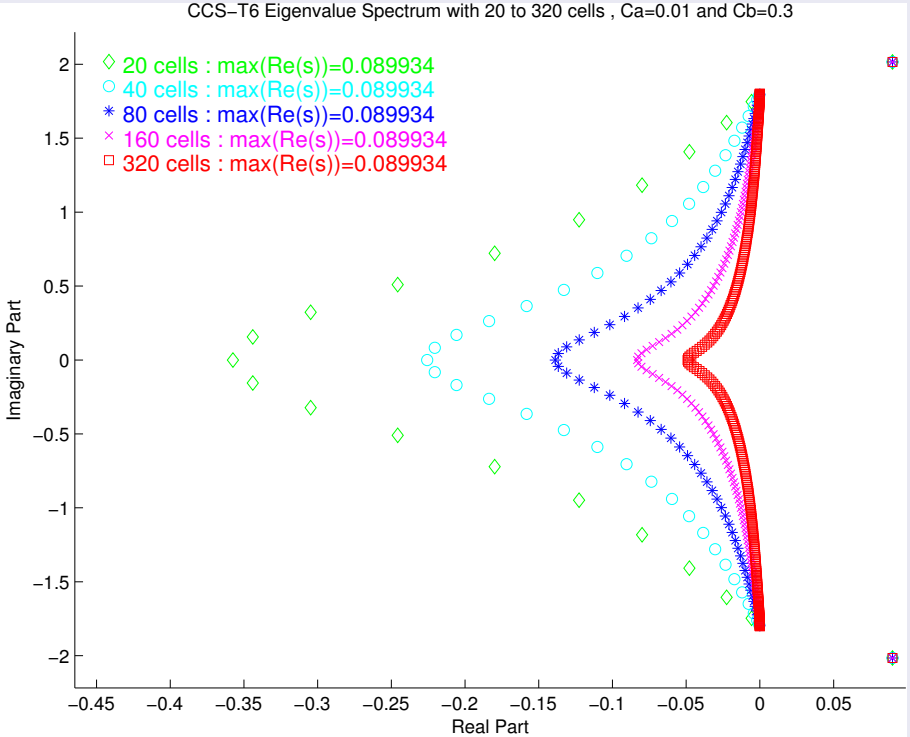
(a) G-K-S analysis



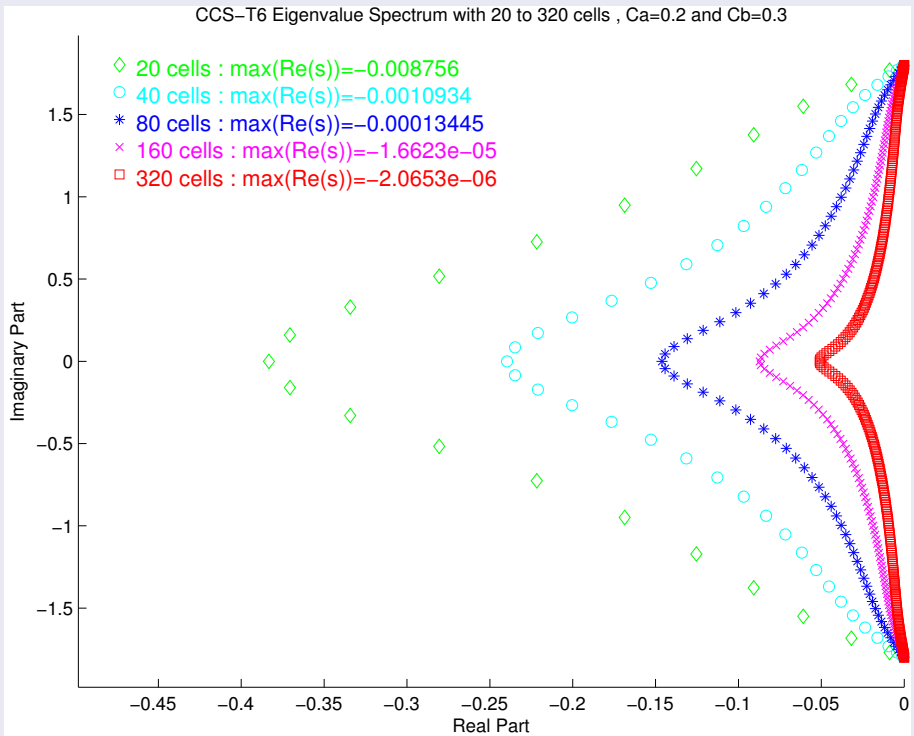
(b) Eigenspectrum analysis

Figure : Real part of the eigenvalues responsible of instability of the CCS-T6 scheme provided with extrapolation and SILW1 boundary conditions.

CCS-T6 provided with extrapolation-SILW2 boundary conditions



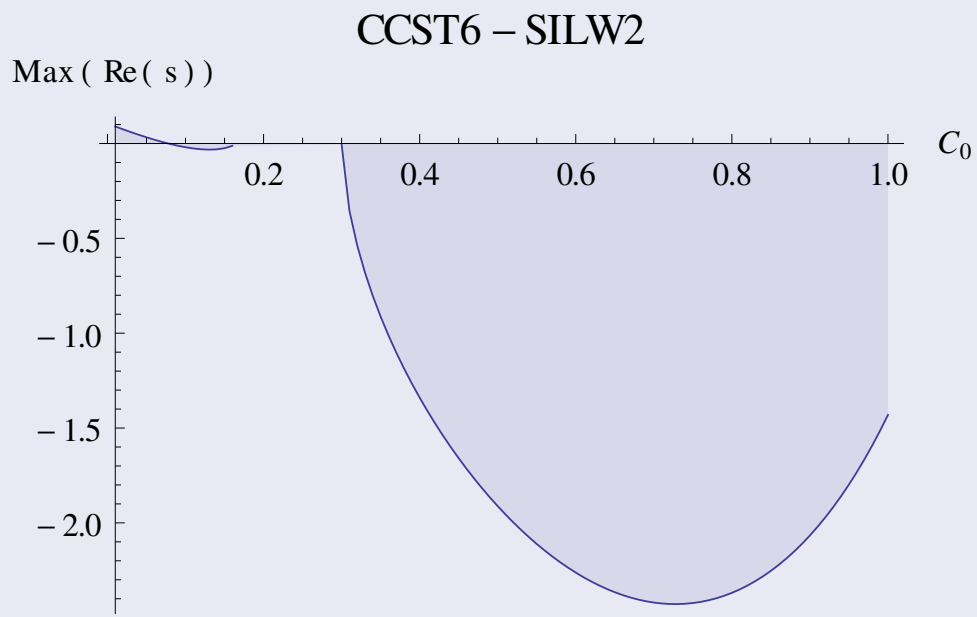
(a) $C_A = 0.01$



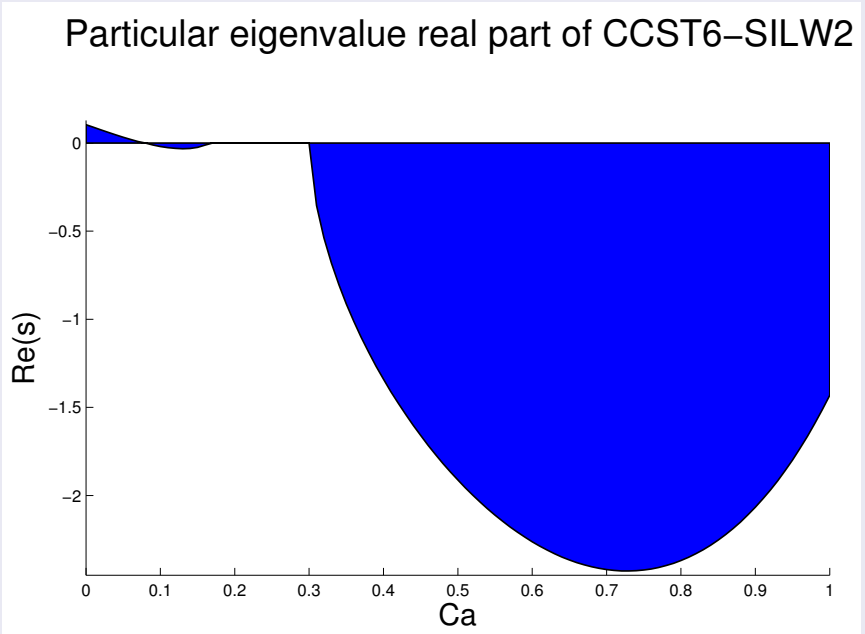
(b) $C_A = 0.2$

Figure : The eigenvalue spectrum of the CCS-T6, provided with SILW procedure with two terms for the inflow boundary, and extrapolation for the outflow boundary.

CCS-T6 provided with extrapolation-SILW2 boundary conditions



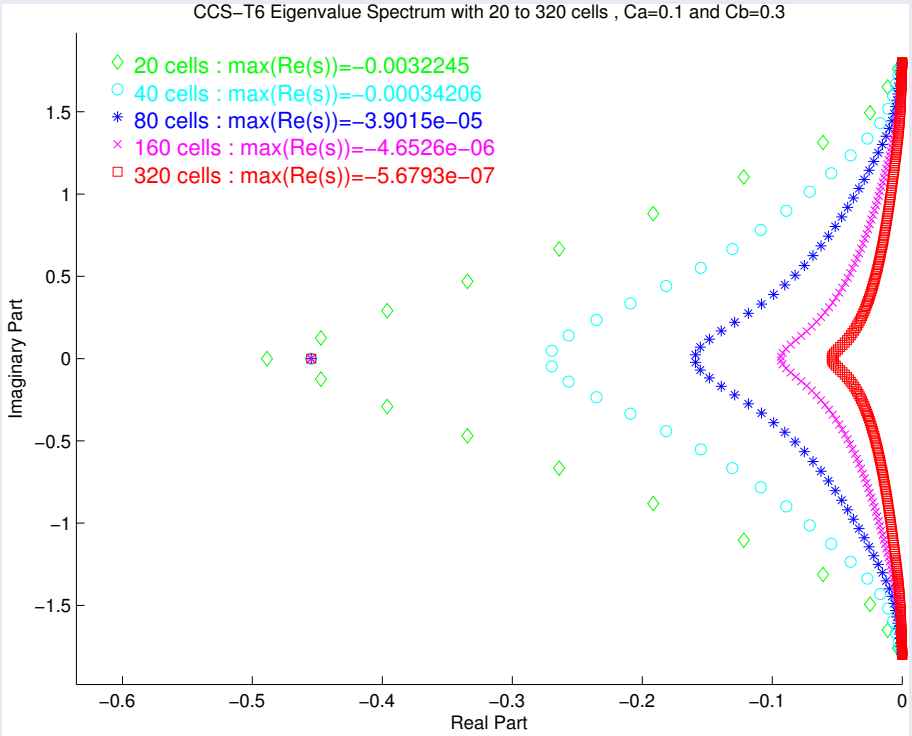
(a) G-K-S analysis



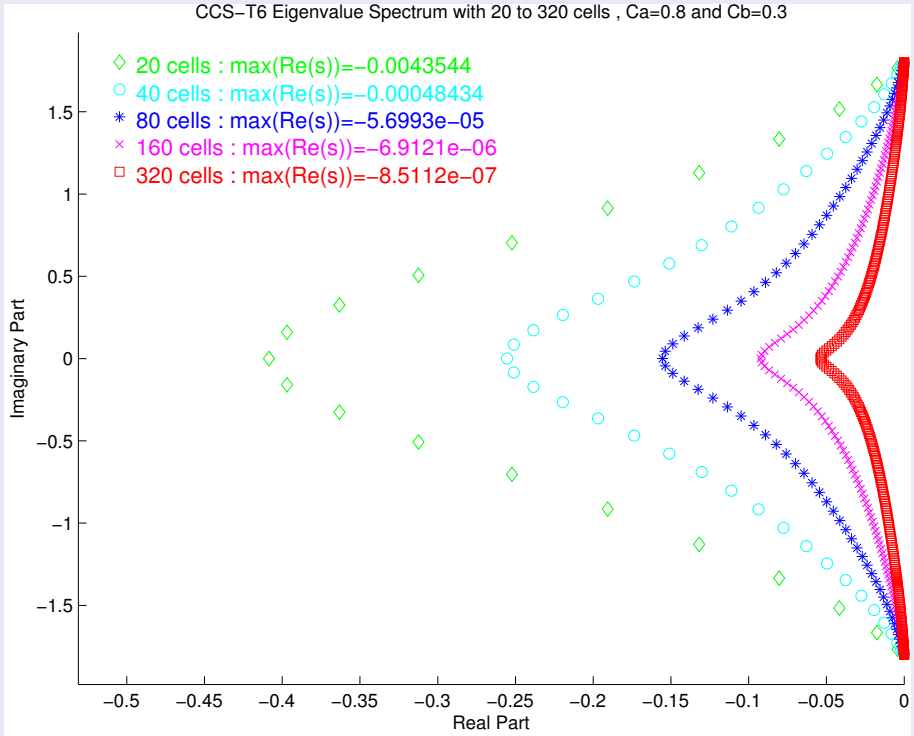
(b) Eigenspectrum analysis

Figure : Real part of the eigenvalues responsible of instability of the CCS-T6 scheme provided with extrapolation and SILW2 boundary conditions.

CCS-T6 provided with extrapolation-SILW3 boundary conditions



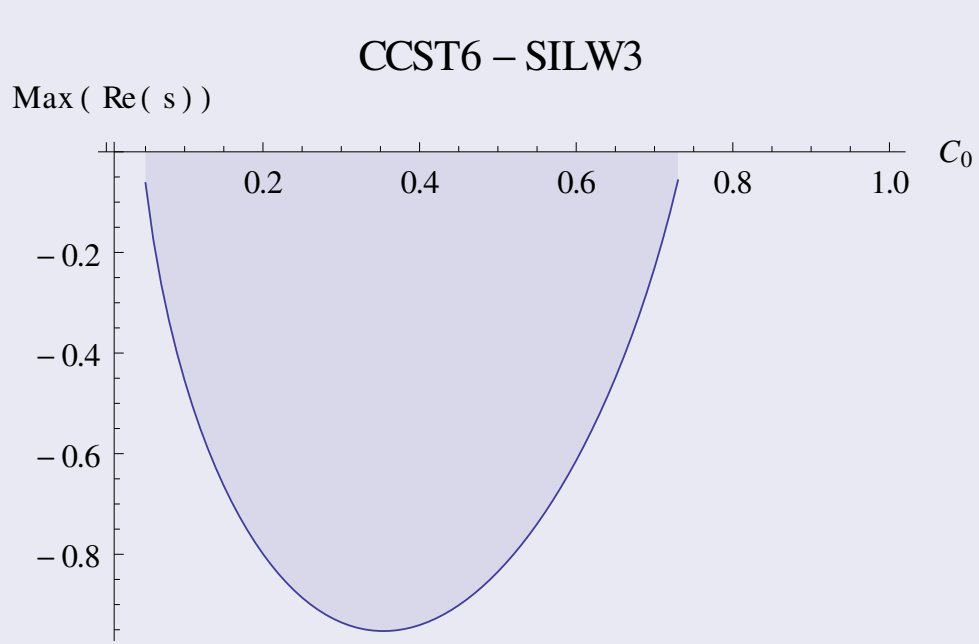
(a) $C_A = 0.1$



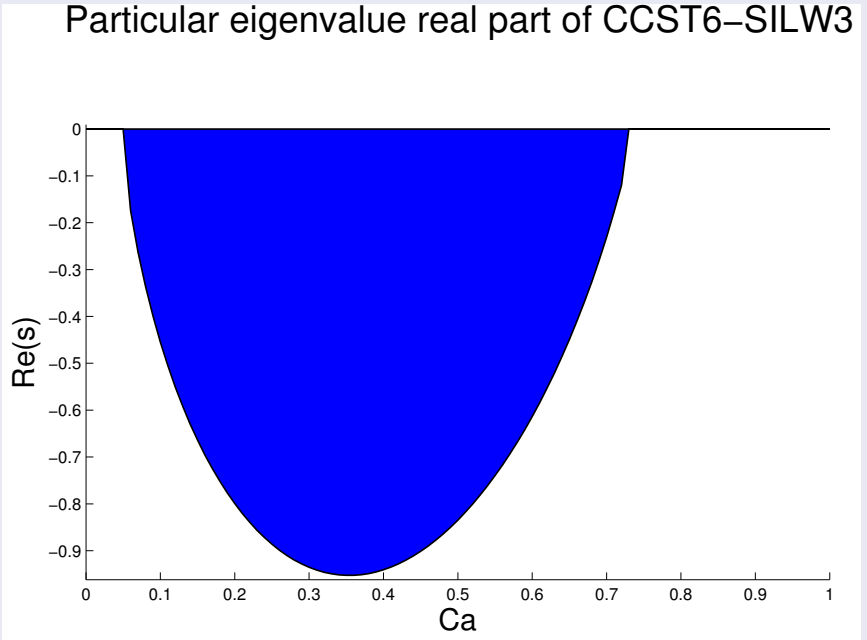
(b) $C_A = 0.8$

Figure : The eigenvalue spectrum of the CCS-T6, provided with SILW procedure with three terms for the inflow boundary, and extrapolation for the outflow boundary.

CCS-T6 provided with extrapolation-SILW3 boundary conditions



(a) G-K-S analysis



(b) Eigenspectrum analysis

Figure : Real part of the eigenvalues responsible of instability of the CCS-T6 scheme provided with extrapolation and SILW3 boundary conditions.

Outcome

- The CCS semidiscrete schemes provided with **extrapolation outflow** boundary condition and **ILW inflow** boundary condition are stable for any CCS and values of C_A and C_B
- The stability of the CCS provided with **extrapolation outflow** boundary condition and **SILW inflow** boundary condition **depends on** the values of C_A and C_B and on the **number of leading terms** k_s

- For some specific value of C_A, the eigenvalue spectrum of CCS provided with extrapolation and simplified inverse Lax-Wendroff boundaries may present some particular eigenvalues independent of the resolution
- These particular eigenvalues correspond to the solution of the eigenvalue problem solved in the G-K-S stability analysis
- Since the extrapolation boundary brings no instability, the two different approaches lead to the same results

Discretization

- We recall the semidiscrete system obtained previously

$$\mathrm{P}\frac{\mathrm{d}\,\boldsymbol{U}}{\mathrm{d}t} = -\frac{A}{\Delta x}\mathrm{Q}\boldsymbol{U}$$

- We apply to this ODE the third-order TVD Runge-Kutta method
- Doing so, the fully discretize problem can be written as

$$\boldsymbol{U}^{n+1} = \mathrm{G}\,\boldsymbol{U}^{n},$$

where $\boldsymbol{U}^n$ and $\boldsymbol{U}^{n+1}$ are the solution vectors at time t^n and t^{n+1}

- The operator G writes

$$\mathrm{G} = \mathrm{I}_d - sgn(A)\frac{|A|\,\Delta t}{\Delta x}\mathrm{P}^{-1}\mathrm{Q} + \left(\frac{|A|\,\Delta t}{\Delta x}\right)^2(\mathrm{P}^{-1}\mathrm{Q})^2 - sgn(A)\left(\frac{|A|\,\Delta t}{\Delta x}\right)^3(\mathrm{P}^{-1}\mathrm{Q})^3,$$

where I_d is the identity matrix

Normal mode analysis

- Assuming a solution of form $u^n = z^n u^0$, the eigenvalue problem writes

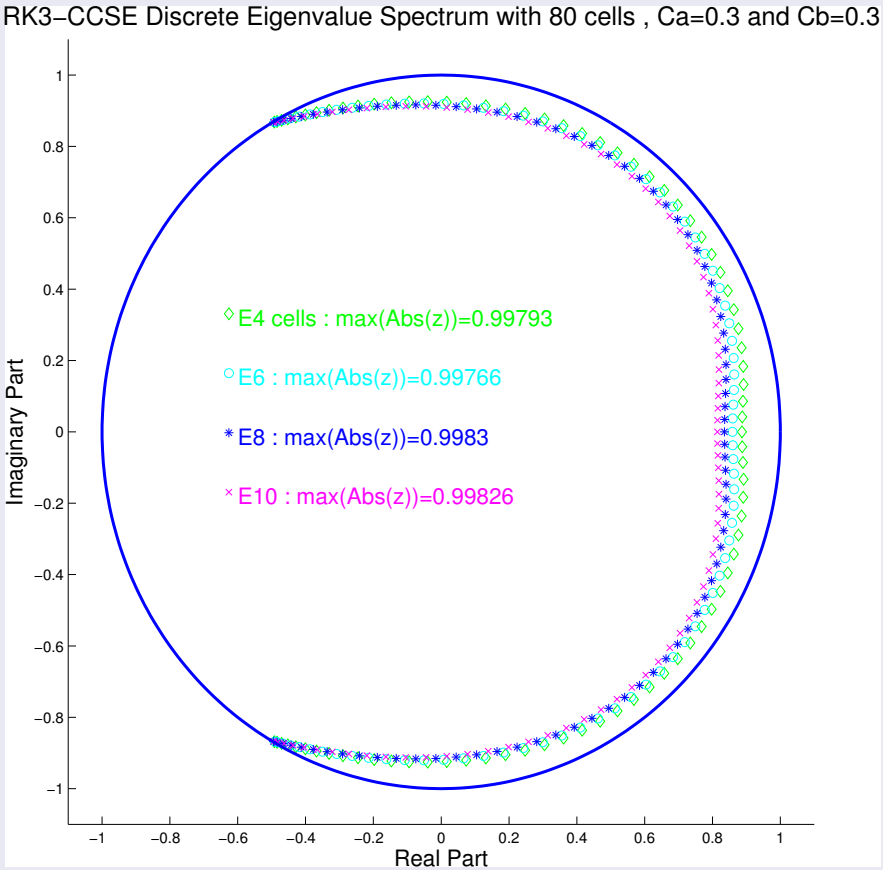
$$z \boldsymbol{U}^n = \mathrm{G}\left(\frac{|A|\,\Delta t}{\Delta x}\right) \boldsymbol{U}^n$$

- We set the condition $CFL = \dfrac{|A|\,\Delta t}{\Delta x}$
- We compute the eigenvalues of matrix $\mathrm{G}(CFL)$
- The fully discrete scheme provided with the considered boundary conditions, on the studied mesh, is stable if the whole eigenvalue spectrum lies in the unit circle

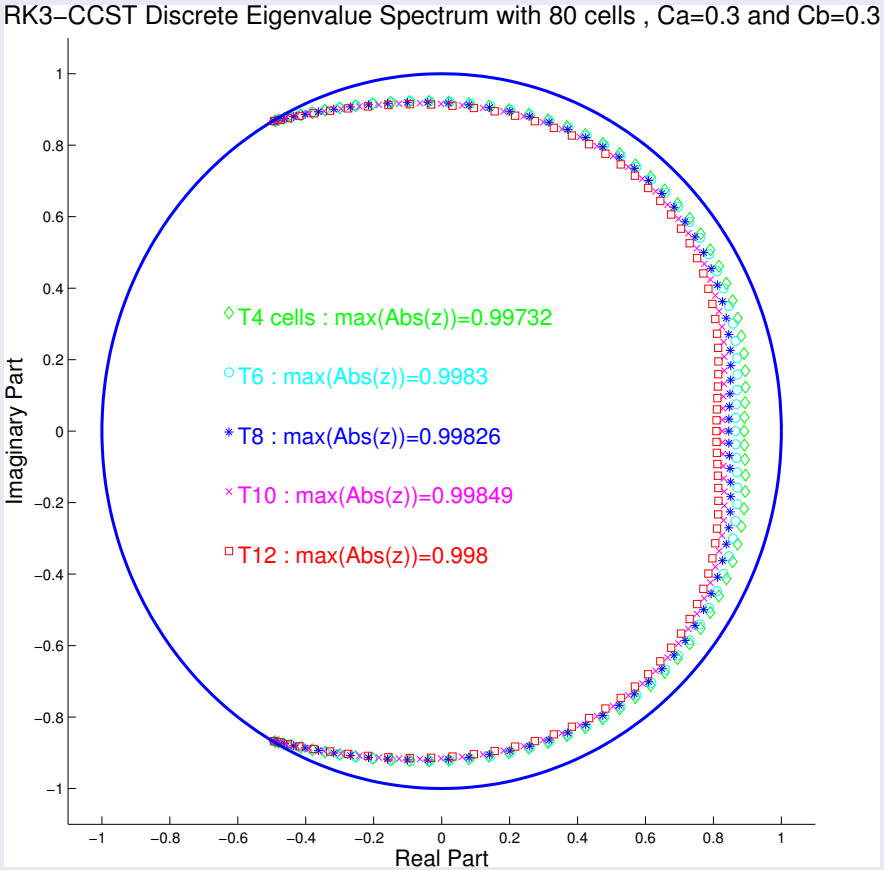
Procedure

- We start the stability analysis with the same CFL condition than for the periodic boundary case
- If the fully discrete scheme is not stable under this CFL ($\exists\, z,\ |z| \geq 1$), we use a decreasing sequence of CFL condition,re-computing at each step the eigenvalues

RK3-CCS provided with extrapolation-ILW boundary conditions



(a) CCS-E



(b) CCS-T

Figure : The eigenvalue spectrum of the RK3-CCS, closed with an Inverse Lax-Wendroff procedure for the inflow boundary, and extrapolation for the outflow boundary, with 40 cells and $C_A = 0.3$ and $C_B = 0.3$.

RK3-CCSP provided with extrapolation-ILW boundary conditions

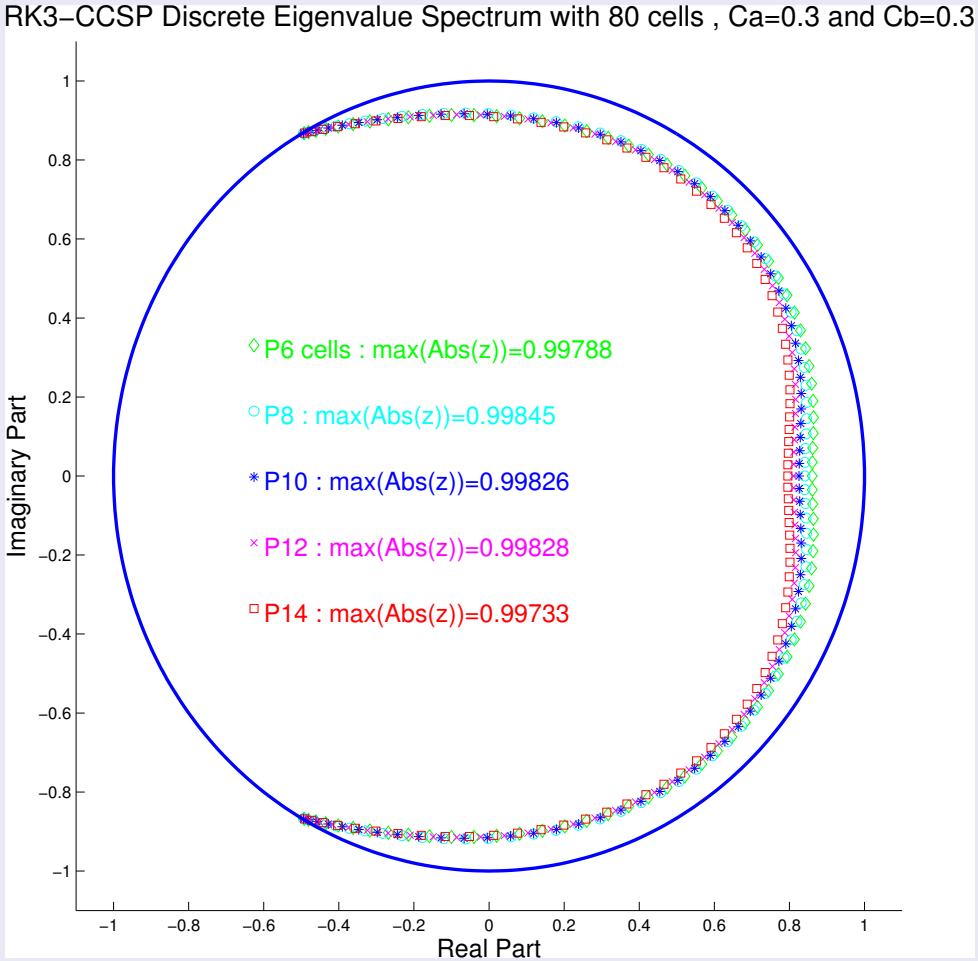


Figure : The eigenvalue spectrum of the RK3-CCSP, closed with an Inverse Lax-Wendroff procedure for the inflow boundary, and extrapolation for the outflow boundary, with 40 cells and $C_A = 0.3$ and $C_B = 0.3$.

RK3-CCST6 with extrapolation-SILW1 boundary conditions

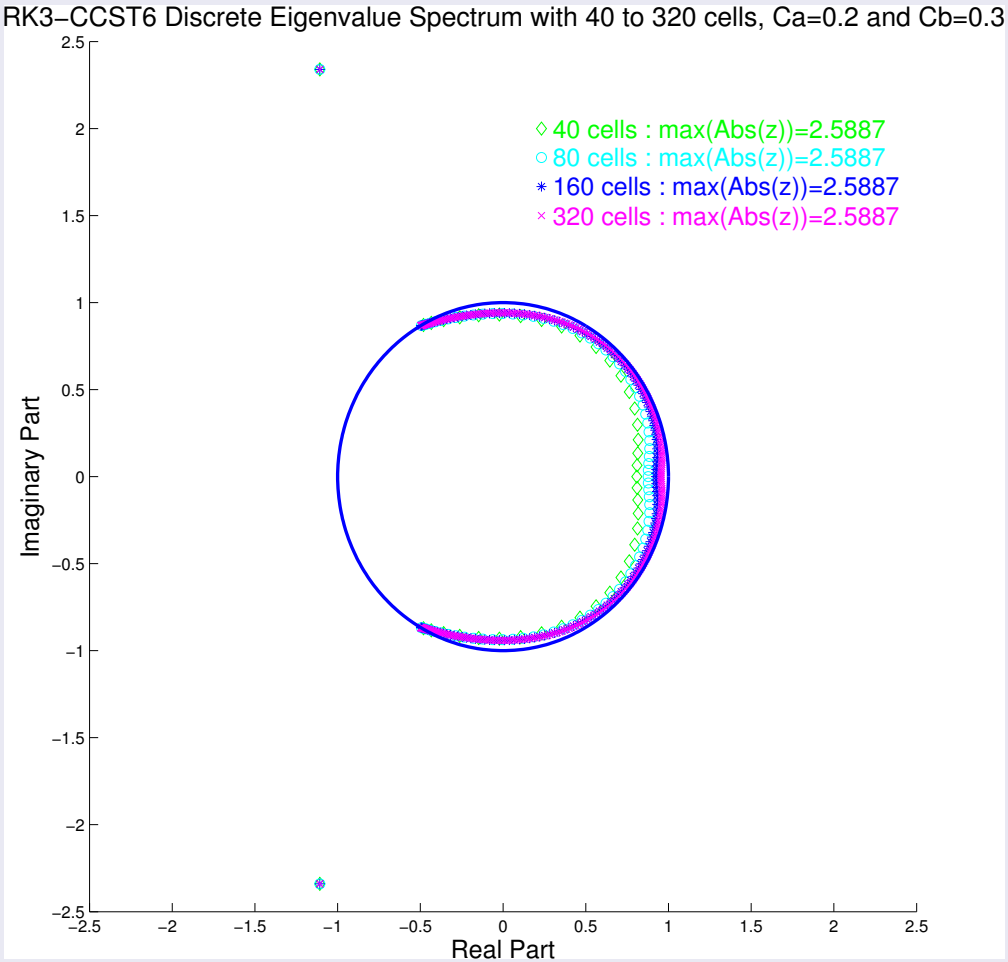
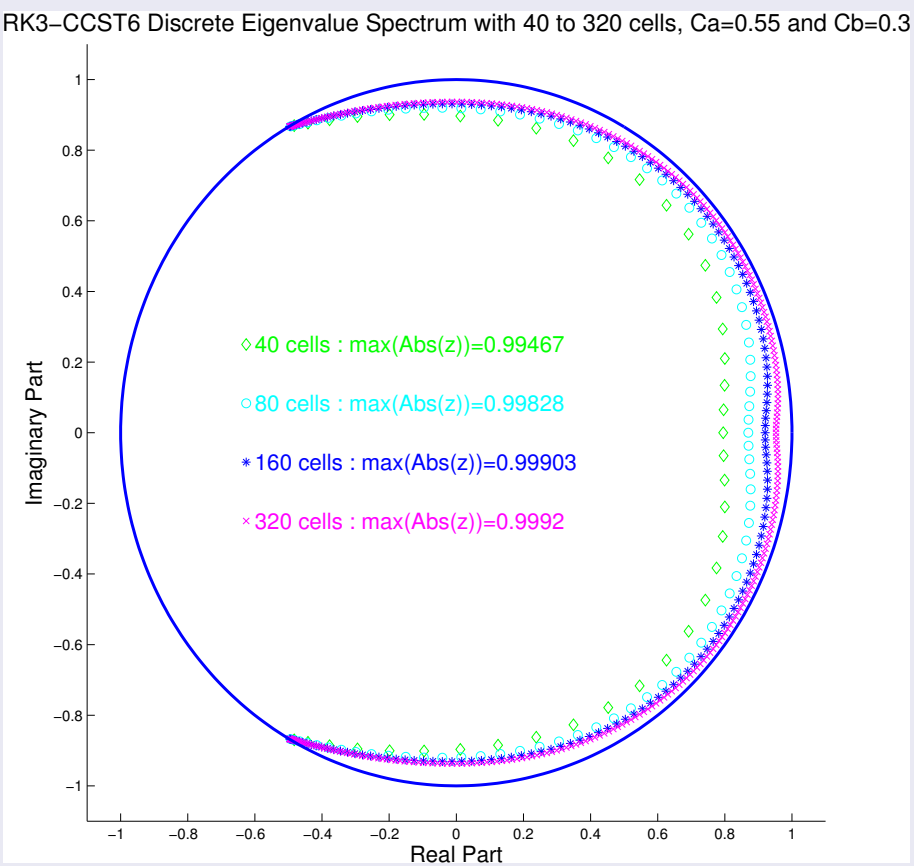
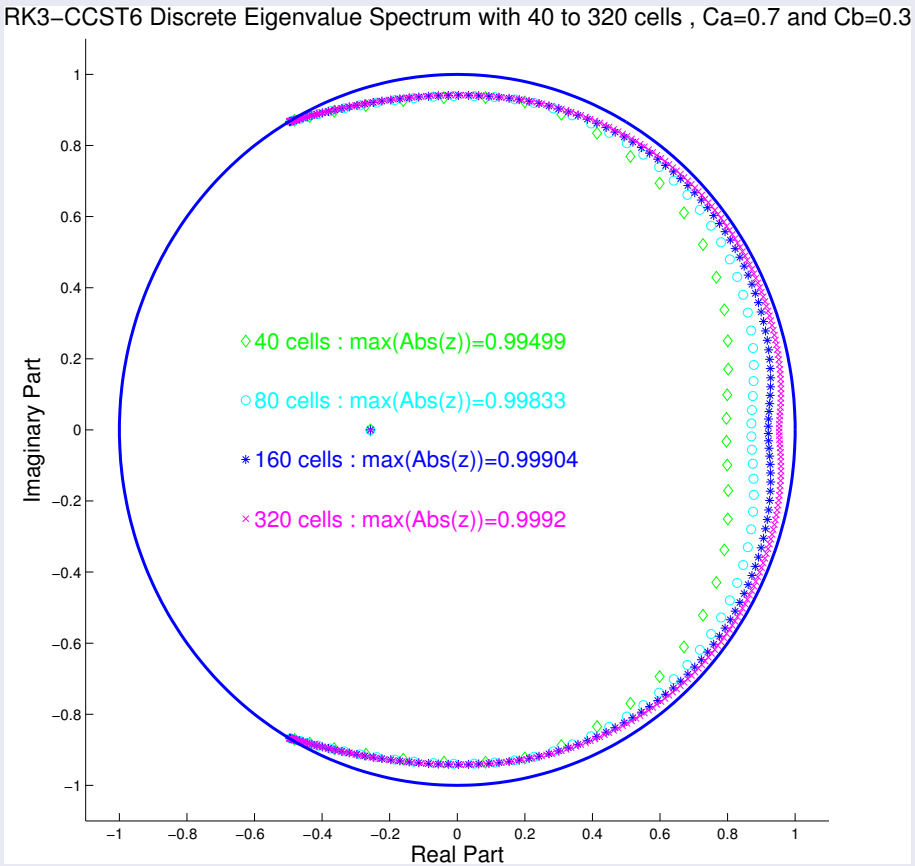


Figure : The eigenvalue spectrum of the RK3-CCST6, provided with SILW procedure with one term for the inflow boundary, and extrapolation for the outflow boundary, with $C_A = 0.2$.

RK3-CCST6 with extrapolation-SILW1 boundary conditions



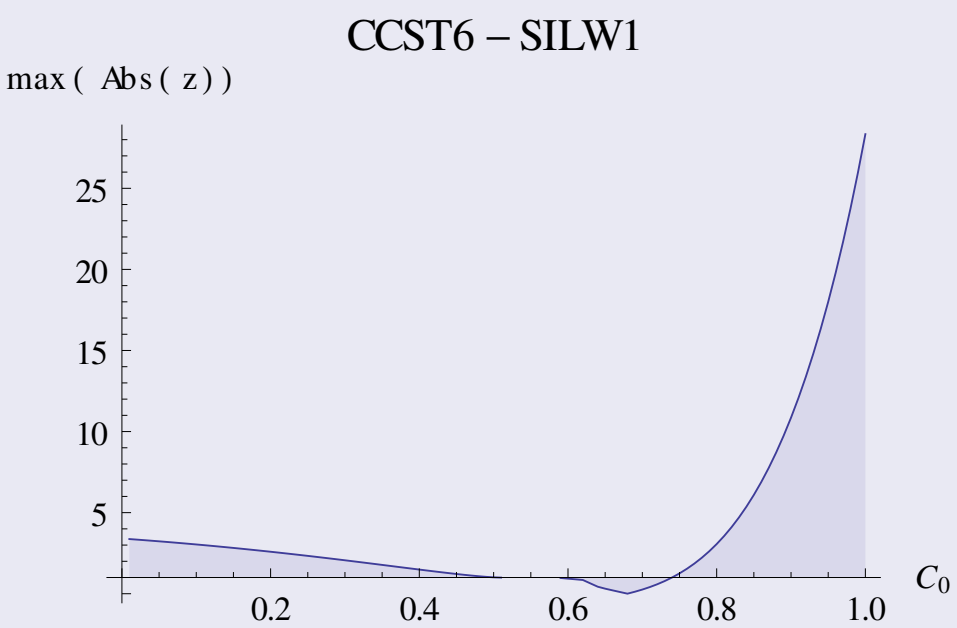
(a) $C_A = 0.55$



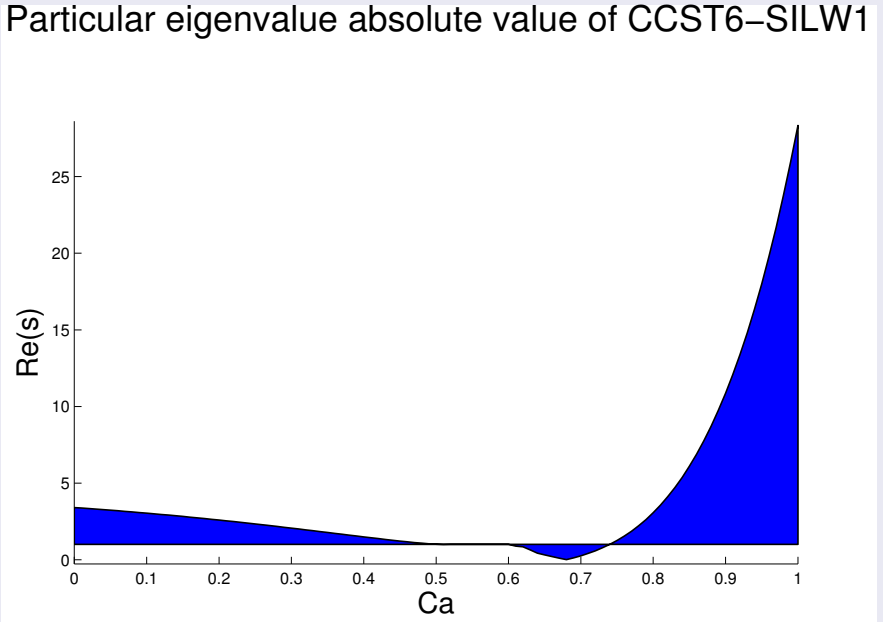
(b) $C_A = 0.7$

Figure : The eigenvalue spectrum of the RK3-CCST6, provided with SILW procedure with one term for the inflow boundary, and extrapolation for the outflow boundary.

RK3-CCST6 with extrapolation-SILW1 boundary conditions



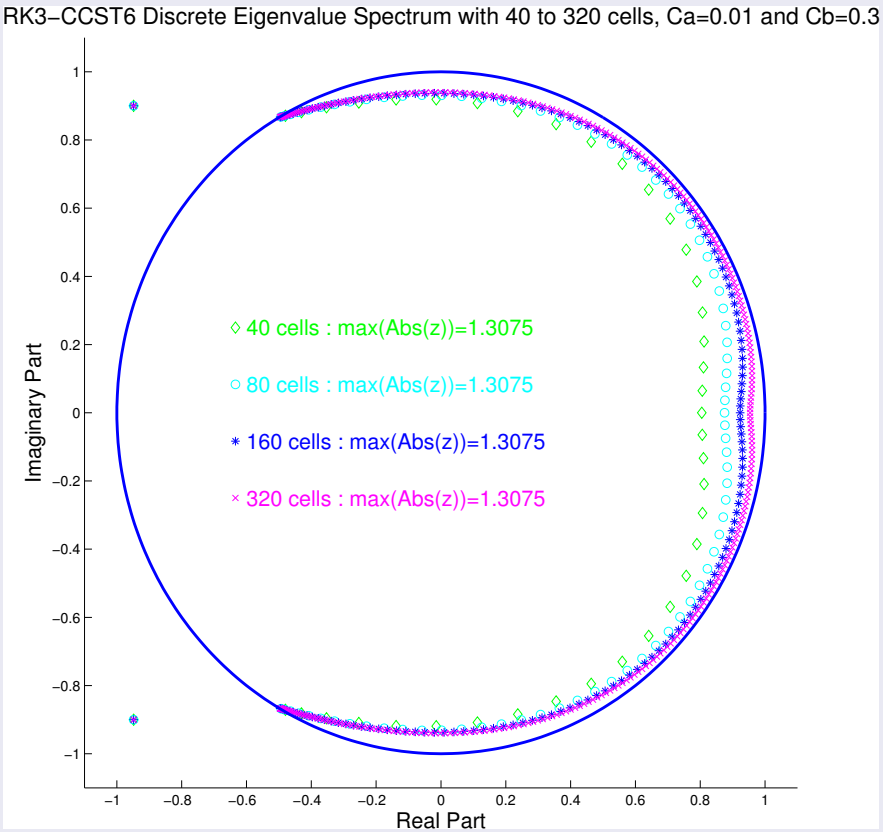
(a) G-K-S analysis



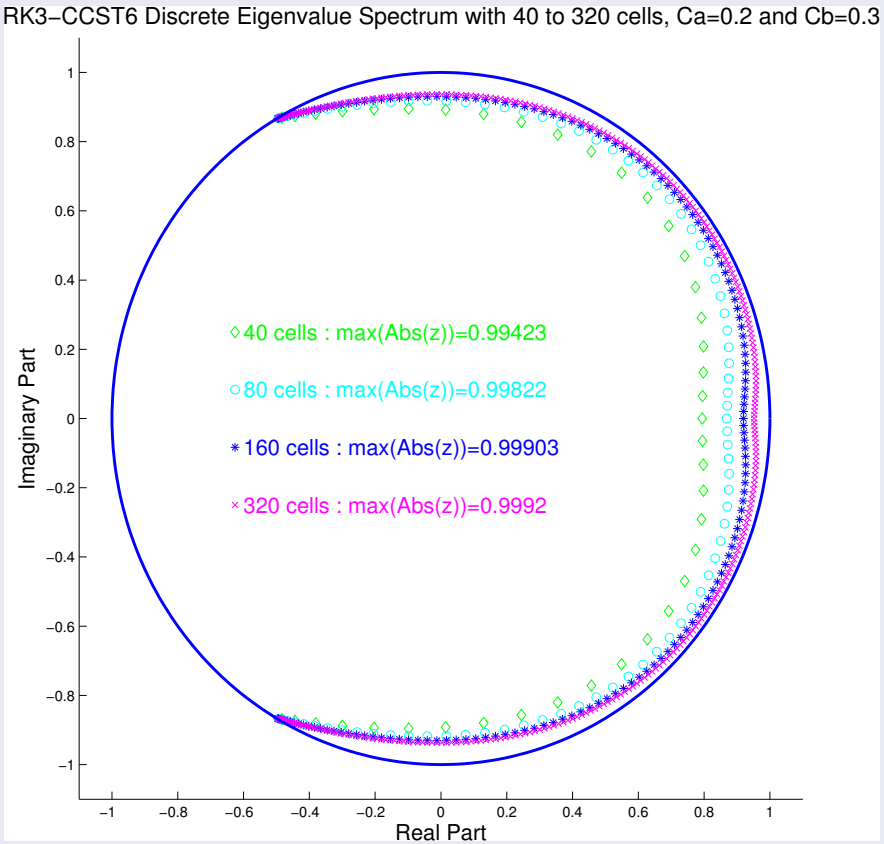
(b) Eigenspectrum analysis

Figure : Real part of the eigenvalues responsible of instability of the RK3-CCST6 scheme provided with extrapolation and SILW1 boundary conditions.

RK3-CCST6 with extrapolation-SILW2 boundary conditions



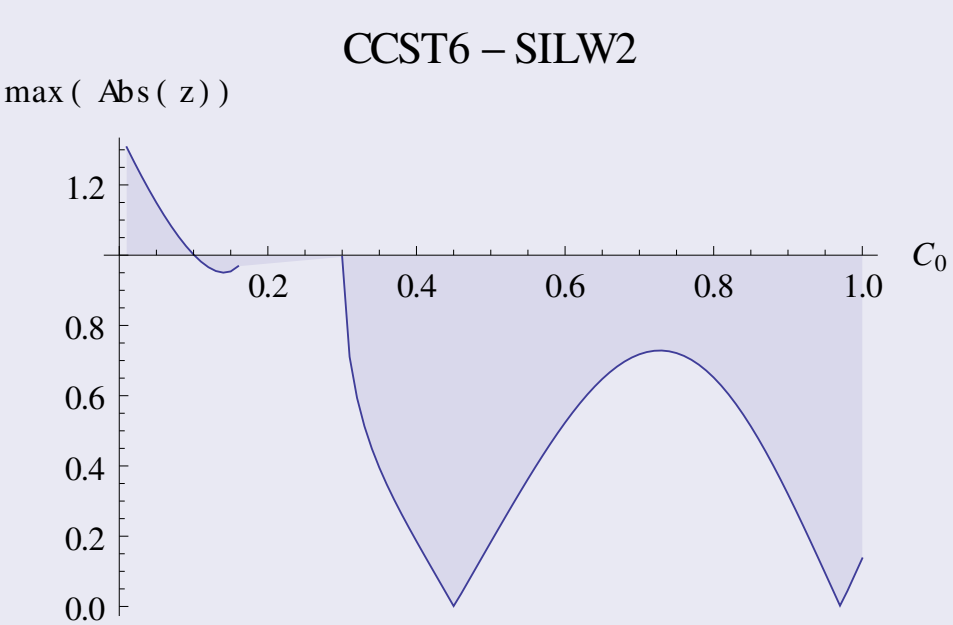
(a) $C_A = 0.01$



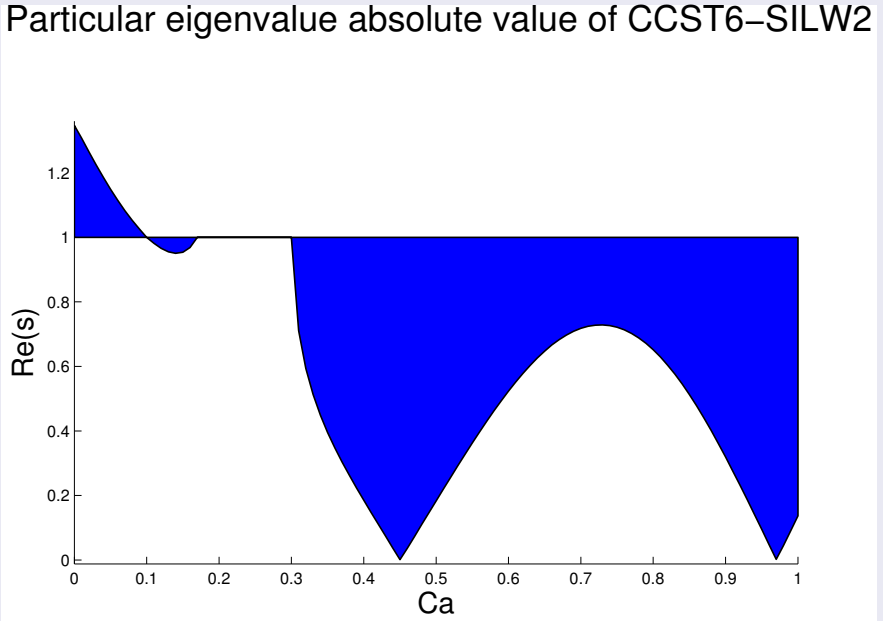
(b) $C_A = 0.2$

Figure : The eigenvalue spectrum of the RK3-CCST6, provided with SILW procedure with two terms for the inflow boundary, and extrapolation for the outflow boundary.

RK3-CCST6 with extrapolation-SILW2 boundary conditions



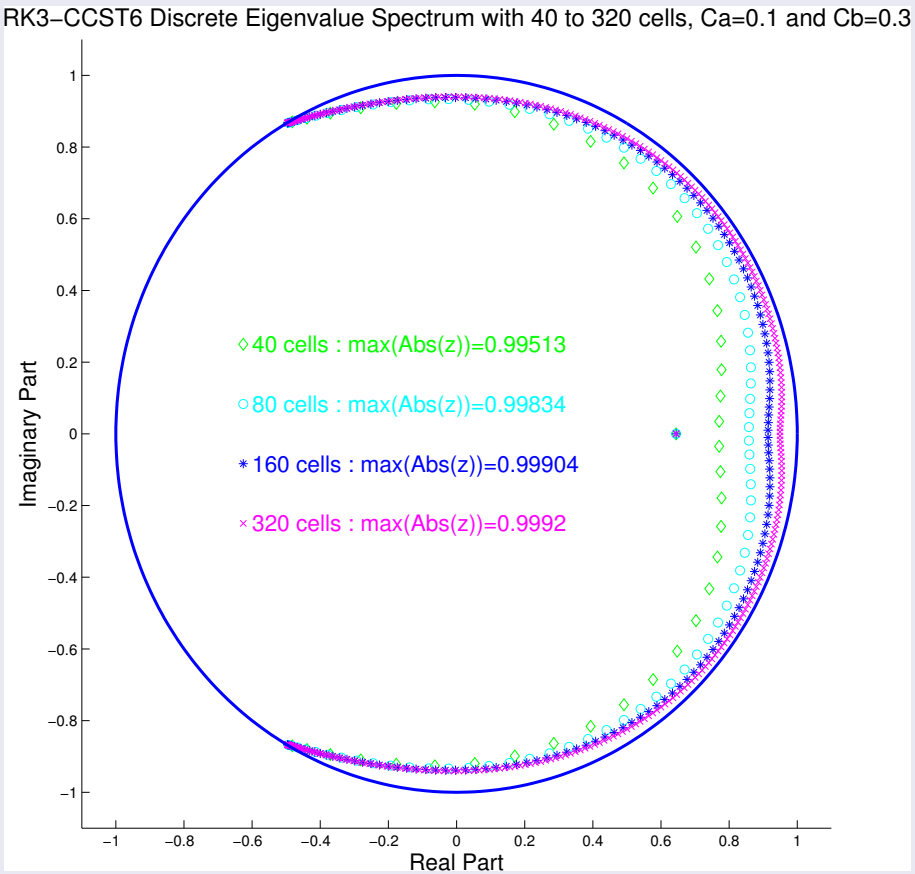
(a) G-K-S analysis



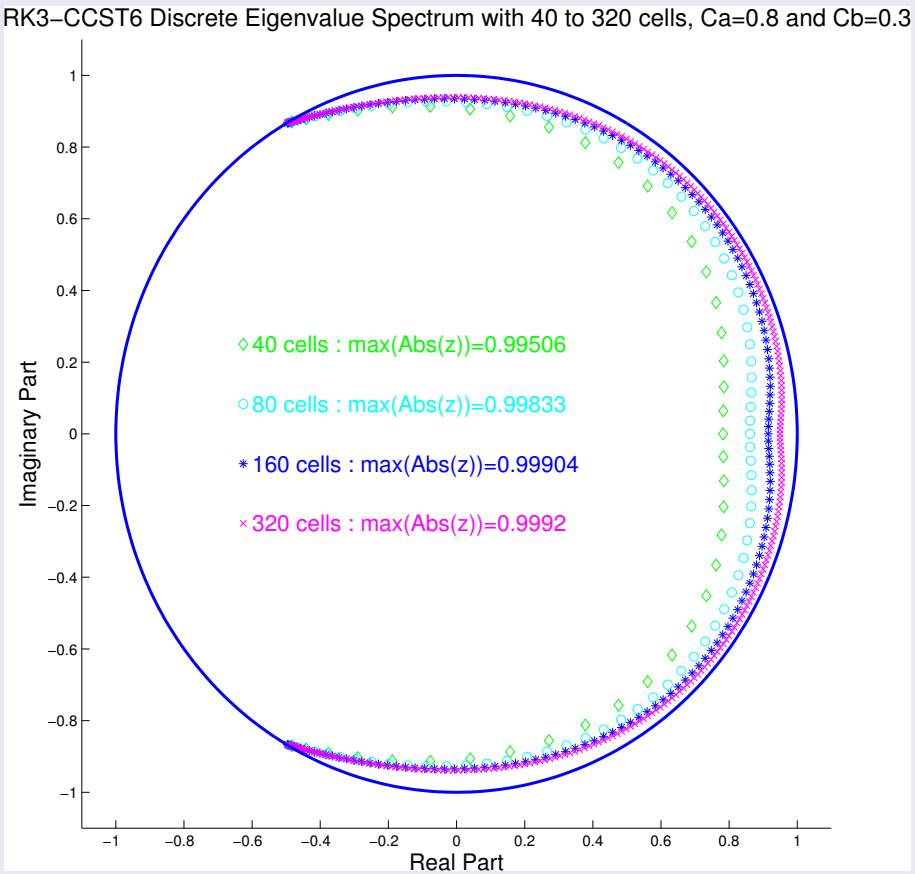
(b) Eigenspectrum analysis

Figure : Real part of the eigenvalues responsible of instability of the RK3-CCST6 scheme provided with extrapolation and SILW2 boundary conditions.

RK3-CCST6 with extrapolation-SILW3 boundary conditions



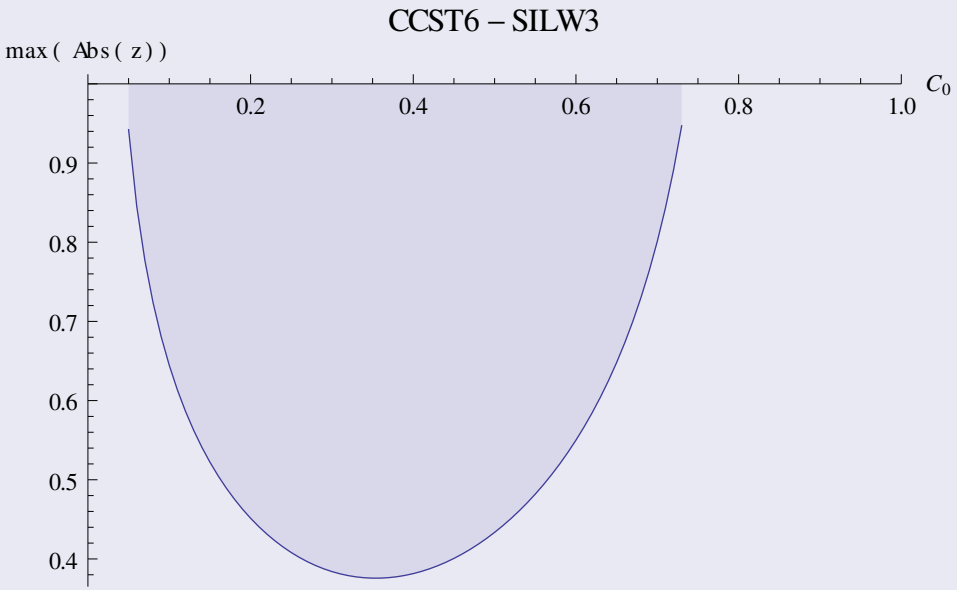
(a) $C_A = 0.1$



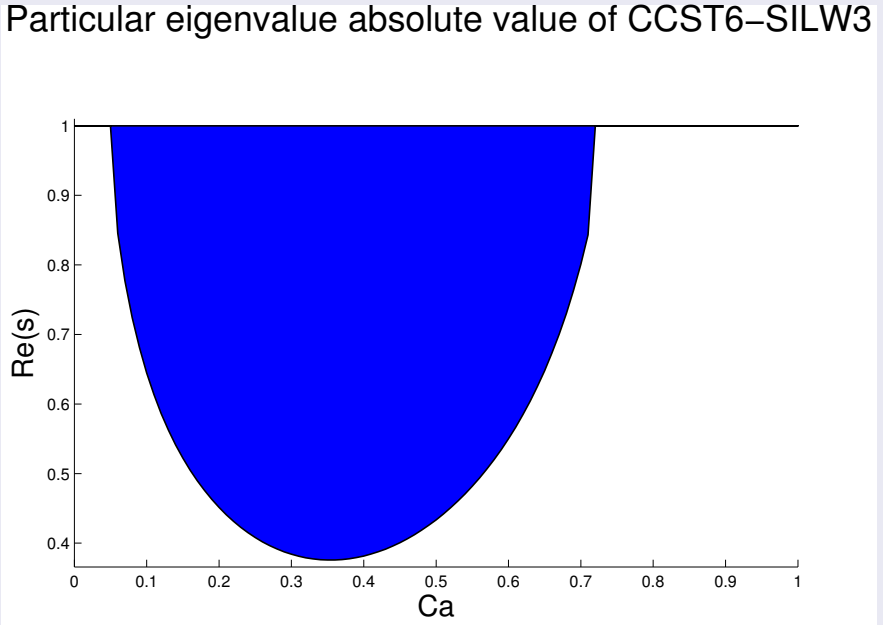
(b) $C_A = 0.8$

Figure : The eigenvalue spectrum of the RK3-CCST6, provided with SILW procedure with three terms for the inflow boundary, and extrapolation for the outflow boundary.

RK3-CCST6 with extrapolation-SILW3 boundary conditions



(a) G-K-S analysis



(b) Eigenspectrum analysis

Figure : Real part of the eigenvalues responsible of instability of the RK3-CCST6 scheme provided with extrapolation and SILW3 boundary conditions.

Outcome

- The RK3-CCS schemes provided with **extrapolation outflow** boundary condition and **ILW inflow** boundary condition are stable for any CCS and values of C_A and C_B
- The stability of the RK3-CCS schemes provided with **extrapolation outflow** boundary condition and **SILW inflow** boundary condition **depends on** the values of C_A and C_B and on the **number of leading terms** k_s

- For some specific value of C_A, the eigenvalue spectrum of RK3-CCS provided with extrapolation and simplified inverse Lax-Wendroff boundaries may present some particular eigenvalues independent of the resolution
- These particular eigenvalues correspond to the solution of the eigenvalue problem solved in the G-K-S stability analysis
- Since the extrapolation boundary brings no oscillations, the two different approaches lead to the same results

Number of terms required in the SILW procedure

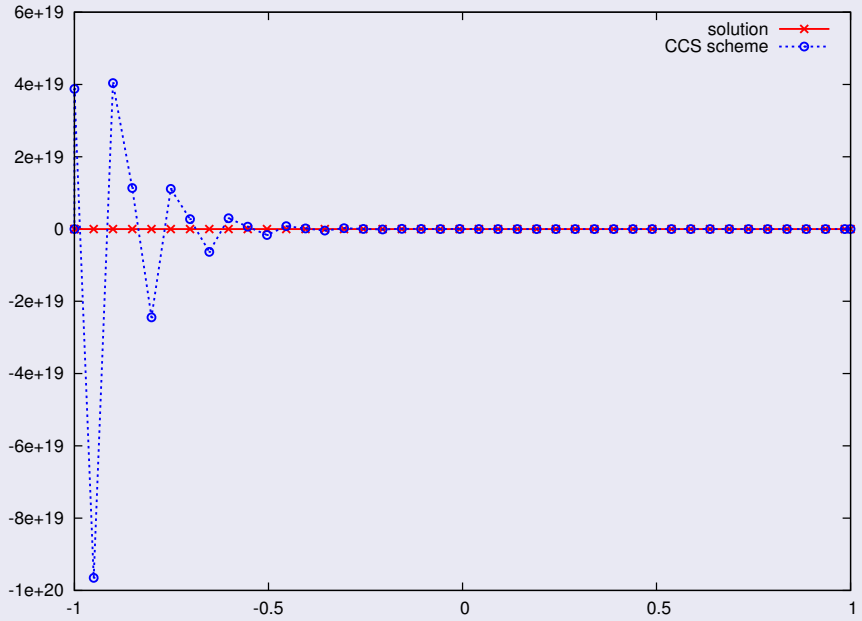
Scheme	Leading terms
CCS-E4	3
CCS-E6	4
CCS-E8	5
CCS-E10	5

Scheme	Leading terms
CCS-T4	3
CCS-T6	3
CCS-T8	5
CCS-T10	8
CCS-T12	9

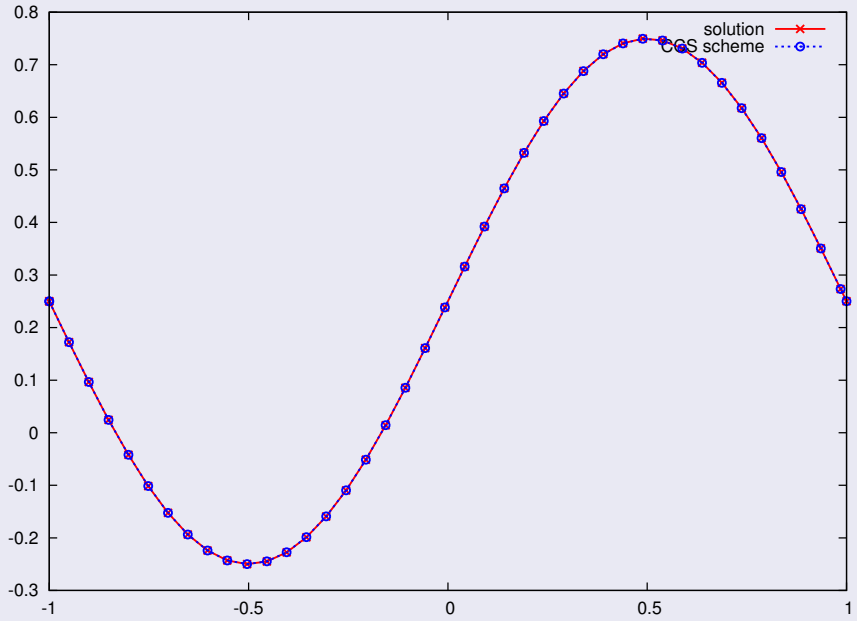
Scheme	Leading terms
CCS-P6	4
CCS-P8	5
CCS-P10	7
CCS-P12	9
CCS-P14	9

Table : Minimum numbers of leading terms required by the different RK3-CCS schemes to remain stable under the same *CFL* than for periodic boundary conditions.

Linear advection case with $C_A = 0.001$ and $C_B = 0.3$



(a) Two leading terms, $t = 10$



(b) Three leading terms, $t = 10000$

Figure : Numerical results obtained with RK3-CCST6 scheme provided with extrapolation and SILW boundaries in the linear advection case ($A = 1$) on $x \in [-1, 1]$, with the initial and boundary condition $u_0(x) = 0.25 + 0.5 \sin(\pi x)$ and $u(-1, t) = 0.25 - 0.5 \sin(\pi (1 + t))$, with 40 cells and $CFL = 0.96$.

Conclusions

- Both G-K-S and eigenspectrum stability analysis have been done, and gave **perfectly consistent results**
- CCS have been proved to remain stable under the same CFL than the periodic boundaries case and for any boundary position, provided with
  - outflow extrapolation boundary
  - inflow inverse Lax-Wendroff boundary
- The number of leading terms required by the SILW procedure for the central compact scheme to remain stable has been determined

Perspectives

- Design an energy stability
- Adapt CCS and ILW to the SBP-SAT operators
- Apply the studied schemes and boundaries to more practical applications (Euler, Navier-Stokes, . . .)

M.H. Carpenter, D. Gottlieb and S. Abarbanel, *Time-stable conditions for finite-difference schemes solving hyperbolic systems: methodology and application to high-order compact schemes.* JCP, 1994.