

Local subcell monolithic DG/FV scheme

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- 2 DG as a subcell FV
- 3 Monolithic subcell DG/FV scheme
- 4 Entropy stabilities
- 5 Maximum principles
- 6 Conclusion?

Scalar Conservation Law (SCL)

- $\partial_t u(\mathbf{x}, t) + \nabla_{\mathbf{x}} \cdot \mathbf{F}(u(\mathbf{x}, t)) = 0, \quad (\mathbf{x}, t) \in \mathbb{R}^d \times [0, T]$
- $u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d$

Fundamental difficulty

- **Even considering smooth flux function $F(\cdot)$ and initial datum $u_0(\cdot)$ (C^∞ for instance), solution may become discontinuous in finite time**
- Standard example: Burgers equation

$$\begin{cases} \partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = 0, & (x, t) \in [0, 1] \times \mathbb{R}^+, \\ u(x, 0) = \sin(2\pi x), & x \in [0, 1]. \end{cases}$$

Strong solution

- Local in time existence and uniqueness of a solution $u \in C^1(\mathbb{R}^d \times [0, t_c[)$

Formation of a discontinuity in finite time

Weak solution

$$\forall \psi \in \mathcal{C}_0^1(\mathbb{R}^d \times \mathbb{R}^+)$$

- $\int \int_{\mathbb{R}^d \times \mathbb{R}^+} \left(u \partial_t \psi + \mathbf{F}(u) \cdot \nabla_x \psi \right) d\mathbf{x} dt - \int_{\mathbb{R}^d} u_0(\mathbf{x}) \psi(\mathbf{x}, 0) d\mathbf{x} = 0$
- Non-uniqueness of a weak solution $u \in L_{loc}^\infty(\mathbb{R}^d \times \mathbb{R}^+)$

Entropy definition

- Let $u \in \mathcal{C}^1(\mathbb{R}^d \times [0, t_c[)$ be a strong solution
- η a strictly convex function is an entropy if $\exists \phi$ s.t.

$$\partial_t \eta(u) + \nabla_x \cdot \phi(u) = 0$$

Unique entropic weak solution

$$\forall \psi \in \mathcal{C}_0^1(\mathbb{R}^d \times \mathbb{R}^+)$$

- Let $u \in L_{loc}^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ be a weak solution
- u is the unique entropic weak solution if it satisfies, for any pair (η, ϕ)

$$\int \int_{\mathbb{R}^d \times \mathbb{R}^+} \left(\eta(u) \partial_t \psi + \phi(u) \cdot \nabla_x \psi \right) d\mathbf{x} dt - \int_{\mathbb{R}^d} \eta(u_0(\mathbf{x})) \psi(\mathbf{x}, 0) d\mathbf{x} \geq 0$$

Formation of a discontinuity in finite time

Challenges of numerical discretization

- **The weak solution we wish to approach may present areas of great regularity as well as discontinuities of great intensity**
 - **Smooth areas**: high accuracy
 - **Discontinuities**: strong robustness and stability
 - **Everywhere**: additional mathematical or physical constraints
 - ↪ Maximum principle or positivity
 - ↪ Entropy inequalities
 - ↪ ...

Numerical schemes and discontinuous approximated solution

- **Finite Volume** **FV**
- (Weighted) Essentially Non-Oscillatory (W)-ENO
- **Discontinuous Galerkin** **DG**

Discontinuous Galerkin scheme

- Introduced by Reed and Hill in 1973 in the frame of the neutron transport
- Major development and improvements by B. Cockburn and C.-W. Shu

Procedure

- Local variational formulation
- Piecewise polynomial approximation of the solution in the cells
- Choice of the numerical fluxes
- Time integration

Advantages

- Natural extension of Finite Volume method
- Excellent analytical properties (L_2 stability, hp -adaptivity, ...)
- Extremely high accuracy (superconvergent for scalar conservation laws)
- Compact stencil (involve only face neighboring cells)

Scalar conservation law

- $\partial_t u(\mathbf{x}, t) + \nabla_x \cdot \mathbf{F}(u(\mathbf{x}, t)) = 0, \quad (\mathbf{x}, t) \in \omega \times [0, T]$
- $u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \omega$

$(k+1)^{\text{th}}$ order semi-discretization

- $\{\omega_c\}_c$ a partition of ω , such that $\omega = \bigcup_c \omega_c$
- $u_h(\mathbf{x}, t)$ the numerical solution, such that $u_h|_{\omega_c} = u_h^c \in \mathbb{P}^k(\omega_c)$

$$u_h^c(\mathbf{x}, t) = \sum_{m=1}^{N_k} u_m^c(t) \sigma_m^c(\mathbf{x})$$

- $\{\sigma_m^c\}_{m=1, \dots, N_k}$ a basis of $\mathbb{P}^k(\omega_c)$, with $N_k = \frac{(k+1)(k+2)}{2}$ in 2D.

Local variational formulation on ω_c

- $\int_{\omega_c} \frac{\partial u_h^c}{\partial t} \psi \, dV = \int_{\omega_c} \mathbf{F}(u_h^c) \cdot \nabla_x \psi \, dV - \int_{\partial \omega_c} \psi \mathcal{F}_n \, dS, \quad \forall \psi \in \mathbb{P}^k(\omega_c)$

- $\mathcal{F}_n = \mathcal{F}(u_h^c, u_h^v, \mathbf{n})$

numerical flux

Numerical example: solid body rotation

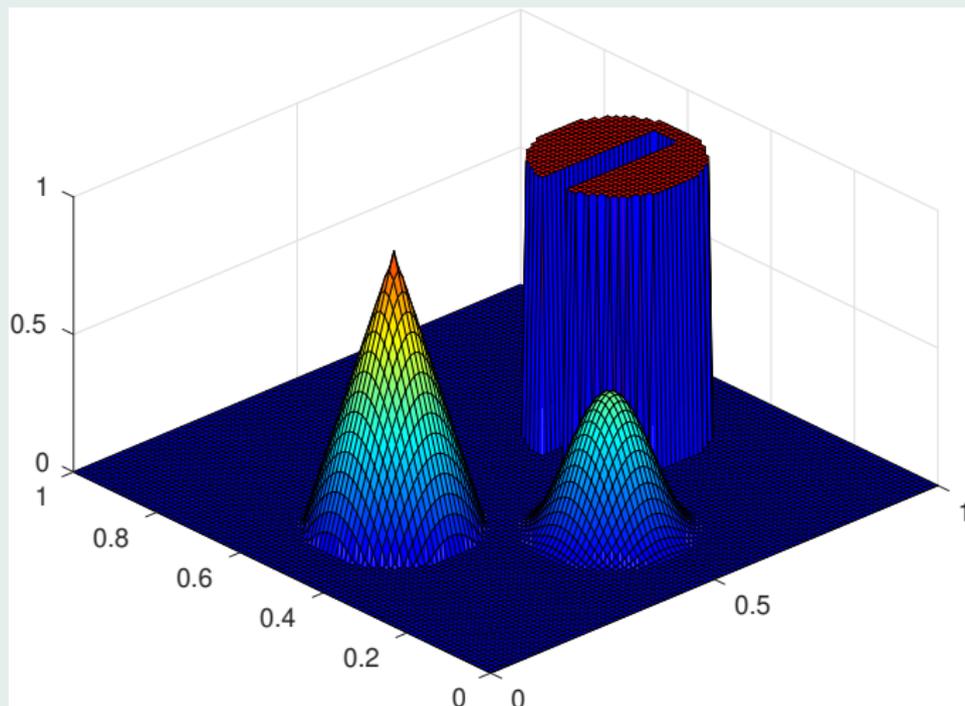
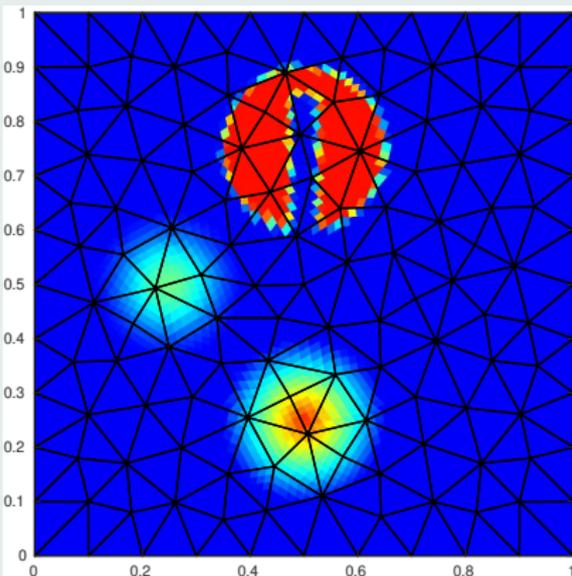
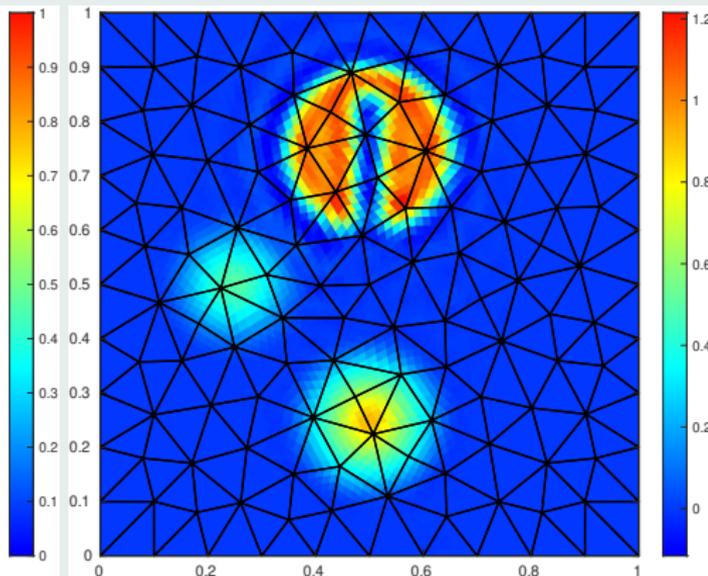


Figure: Rotation of composite signal: initial solution

Rotation of composite signal



(a) Initial solution



(b) Solution after one rotation

Figure: 6th-order DG scheme on 242 cells

Rotation of composite signal

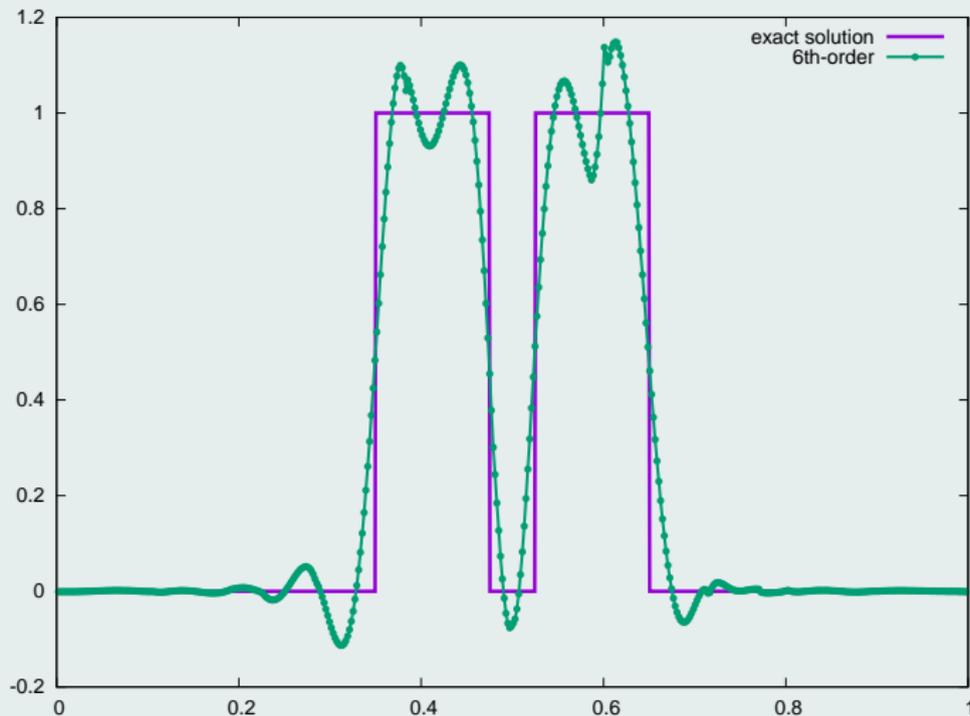
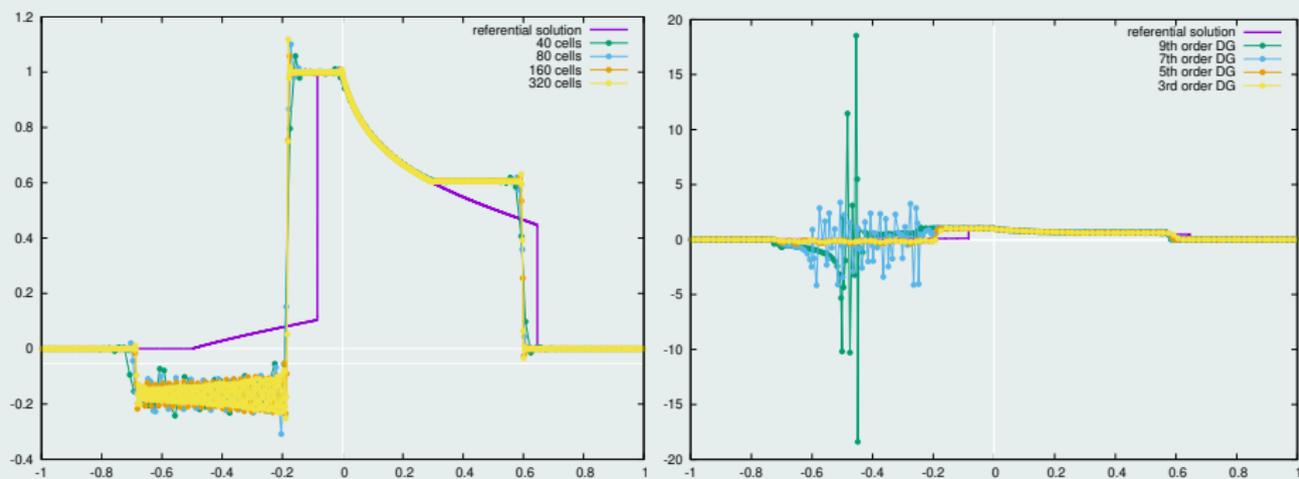


Figure: 6th-order DG scheme on 242 cells: profiles for $y = 0.75$

Spurious oscillations, aliasing and non-entropic behavior



(a) Non-entropic behavior

(b) Aliasing phenomenon

Figure : DG solutions for the Buckley non-convex flux case

Gibbs phenomenon and non-admissible solution

- High-order schemes leads to spurious oscillations near discontinuities
- Non-admissible solution potentially leading to a crash
- Vast literature of how prevent this phenomenon to happen:

⇒ *a priori* and *a posteriori* limitations

A priori limitation

- Artificial viscosity
- Slope/moment/hierarchical limiter
- ENO/WENO limiter
- Flux limiter and FCT schemes
- **Monolithic HO/LO schemes**

A posteriori limitation

- MOOD (“Multi-dimensional Optimal Order Detection”)
- A posteriori subcell correction

Admissible numerical solution

- Maximum principle / positivity preserving
- Limit the apparition of spurious oscillations
- Ensure a correct entropic behavior

Preserving high-accuracy and subcell resolution

Reduce the characteristic length of action

Methodology

Blend, at the subcell scale, high-order DG and 1st-order FV



F.V., *A Posteriori Correction of High-Order DG Scheme through Subcell Finite Volume Formulation and Flux Reconstruction*. JCP, 2018.



F.V., *Local subcell monolithic DG/FV convex property preserving scheme on unstructured grids and entropy consideration*. JCP, 2024.

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DG as a subcell Finite Volume

- Rewrite DG scheme as a FV-like scheme on a subgrid

Cell subdivision into N_S subcells

$$N_S \geq N_k \equiv \#\mathbb{P}^k$$

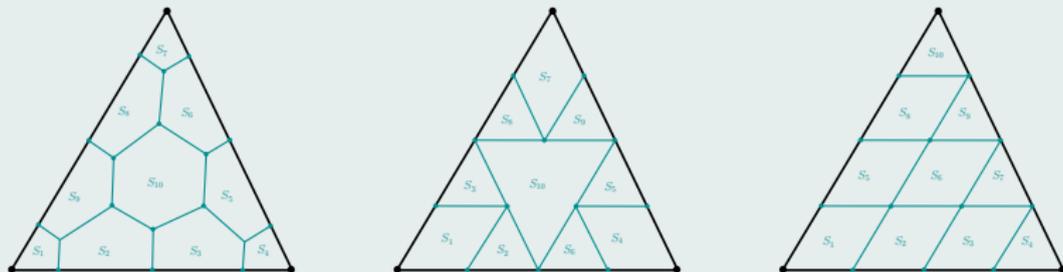


Figure : Examples of $N_S = N_k$ subdivision for \mathbb{P}^3 DG scheme on a triangle

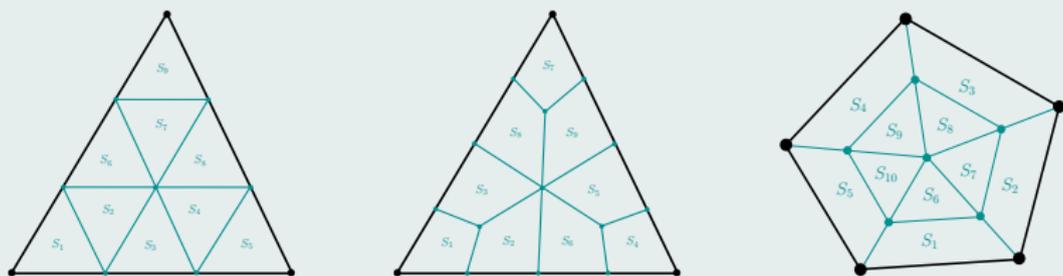


Figure : Examples of $N_S \geq N_k$ subdivision

DG schemes through residuals

$$\bullet \sum_{m=1}^{N_k} \frac{d u_m^c}{dt} \int_{\omega_c} \sigma_m \sigma_p dV = \oint_{\omega_c} \mathbf{F}(u_h^c) \cdot \nabla_x \sigma_p dV - \oint_{\partial\omega_c} \sigma_p \mathcal{F}_n dS, \quad \forall p \in \llbracket 1, N_k \rrbracket$$

 \implies

$$M_c \frac{dU_c}{dt} = \Phi_c$$

- $\bullet (U_c)_m = u_m^c$ Solution moments
- $\bullet (M_c)_{mp} = \int_{\omega_c} \sigma_m \sigma_p dV$ Mass matrix
- $\bullet (\Phi_c)_m = \oint_{\omega_c} \mathbf{F}(u_h^c) \cdot \nabla_x \sigma_m dV - \oint_{\partial\omega_c} \sigma_m \mathcal{F}_n dS$ DG residuals

Subdivision and definition

- $\bullet \omega_c$ is subdivided into N_s subcells S_m^c
- \bullet Let us define $\bar{\psi}_m^c = \frac{1}{|S_m^c|} \int_{S_m^c} \psi dV$ the subcell mean value

Submean values

$$\bullet \bar{u}_m^c = \frac{1}{|S_m^c|} \sum_{q=1}^{N_k} u_q^c \int_{S_m^c} \sigma_q dV \quad \Rightarrow \quad \boxed{\bar{U}_c = P_c U_c}$$

$$\bullet (\bar{U}_c)_m = \bar{u}_m^c \quad \text{Submean values}$$

$$\bullet (P_c)_{mp} = \frac{1}{|S_m^c|} \int_{S_m^c} \sigma_p dV \quad \text{Projection matrix}$$

$$\Rightarrow \quad \boxed{\frac{d\bar{U}_c}{dt} = P_c M_c^{-1} \Phi_c}$$

Admissibility of the cell sub-partition into subcells

$$\bullet P_c^t P_c \quad \text{has to be non-singular}$$

$$\Rightarrow \quad \boxed{U_c = (P_c^t P_c)^{-1} P_c^t \bar{U}_c} \quad \text{Least square procedure}$$

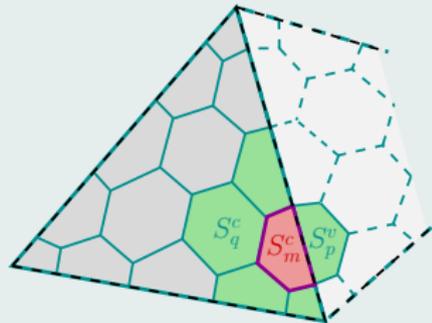
$$\bullet \text{ If } N_s = N_k, \quad \bar{U}_c = P_c U_c \iff U_c = P_c^{-1} \bar{U}_c$$

Subcell Finite Volume: reconstructed fluxes

- Let us introduce the **reconstructed fluxes**

$$\frac{d\bar{u}_m^c}{dt} = -\frac{1}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} I_{mp} \widehat{F}_{pm}$$

- \mathcal{V}_m^c the set of face neighboring subcells of S_m^c
- We impose that on the boundary of cell ω_c , so for $S_p^v \notin \omega_c$



$$I_{mp} \widehat{F}_{pm} = \oint_{f_{mp}^c} \mathcal{F}_n dS \equiv \oint_{f_{mp}^c} \mathcal{F}(u_h^c, u_h^v, \mathbf{n}_{mp}^c) dS$$

- $\widetilde{\mathcal{V}}_m^c$ the set of face neighboring subcells of S_m^c belonging to ω_c
- Let A_c be the adjacency matrix such that

$$(A_c)_{mp} = \begin{cases} 1 & \text{if } S_p^v \in \widetilde{\mathcal{V}}_m^c \text{ with } m < p, \\ -1 & \text{if } S_p^v \in \widetilde{\mathcal{V}}_m^c \text{ with } m > p, \\ 0 & \text{if } S_p^v \notin \widetilde{\mathcal{V}}_m^c. \end{cases}$$

Subcell Finite Volume: reconstructed fluxes

- Let us introduce $D_c = \text{diag}(|S_1^c|, \dots, |S_{N_k}^c|)$ and $(B_c)_m = \int_{\partial S_m^c \cap \partial \omega_c} \mathcal{F}_n \, dS$
- Let \widehat{F}_c be the vector containing all the interior faces reconstructed fluxes

$$-A_c \widehat{F}_c = D_c P_c M_c^{-1} \phi_c + B_c$$

Graph Laplacian technique

- $A_c \in \mathcal{M}_{N_s \times N_f^c}$ with N_f^c the number of interior faces

$$(L_c)_{mp} := (A_c A_c^t)_{mp} = \begin{cases} |\widetilde{\mathcal{V}}_m^c| & \text{if } m = p, \\ -1 & \text{if } S_p^v \in \widetilde{\mathcal{V}}_m^c, \\ 0 & \text{otherwise.} \end{cases}$$

- $L_c \mathbf{1} = \mathbf{0}$ where $\mathbf{1} = (1, \dots, 1)^t \in \mathbb{R}^{N_s}$

- $\Pi = \frac{1}{N_s} (\mathbf{1} \otimes \mathbf{1}) \in \mathcal{M}_{N_s}$

Graph Laplacian technique

- Let \mathcal{L}_c^{-1} be the pseudo-inverse of L_c such that

$$\mathcal{L}_c^{-1} = (L_c + \lambda \Pi)^{-1} - \frac{1}{\lambda} \Pi \quad \forall \lambda \neq 0$$

- Then, \widehat{F}_c is uniquely defined as following

$$\widehat{F}_c = -A_c^t \mathcal{L}_c^{-1} (D_c P_c M_c^{-1} \Phi_c + B_c)$$

- The only terms depending on the time are Φ_c and B_c
- Equivalently, the polynomial solution governing equation is given by

$$\frac{dU_c}{dt} = -P_c^{-1} D_c^{-1} (A_c \widehat{F}_c + B_c)$$

remark

- This unique solution does exist since $(D_c P_c M_c^{-1} \Phi_c + B_c) \cdot \mathbf{1} = 0$

Different cell subdivisions

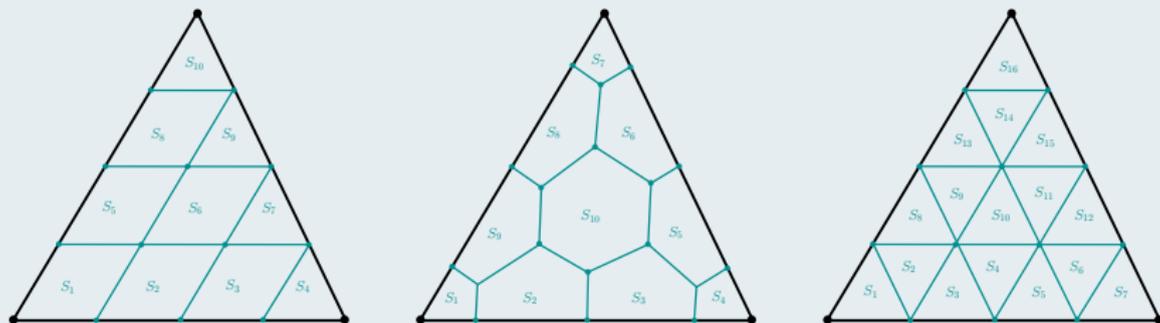
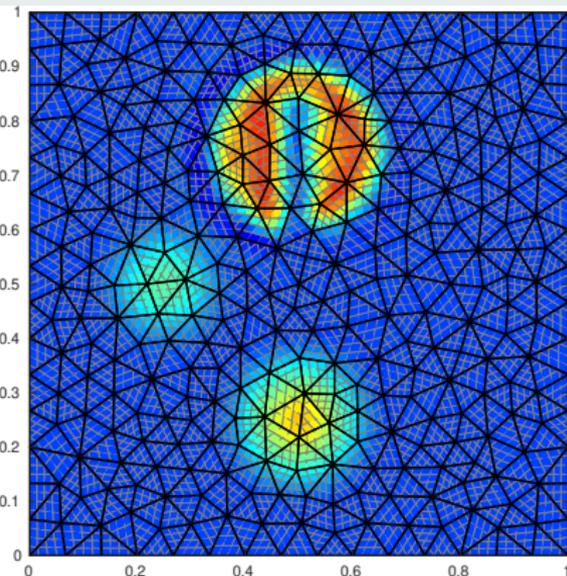


Figure: Examples of easily generalizable subdivisions for a triangle cell

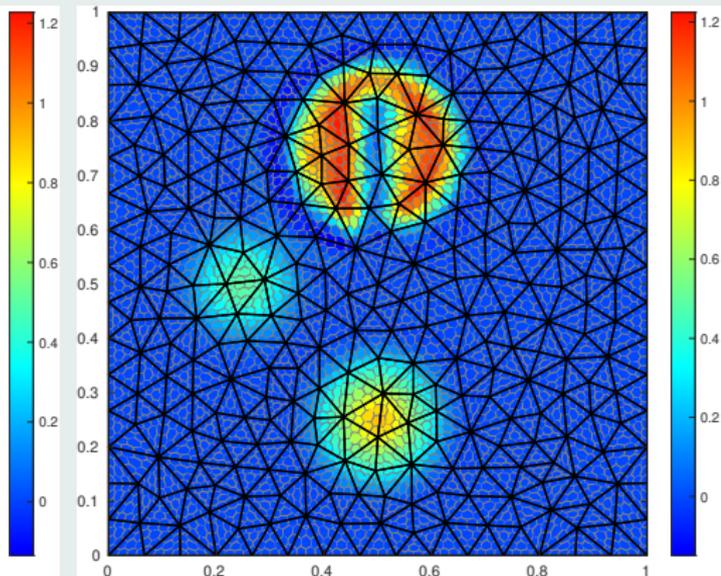
DG is DG

- Only the functional space matters
- The cell subdivision has no influence on the resulting scheme
- Even in the case where $N_s > N_k$

Rotation of a composite signal after one full rotation



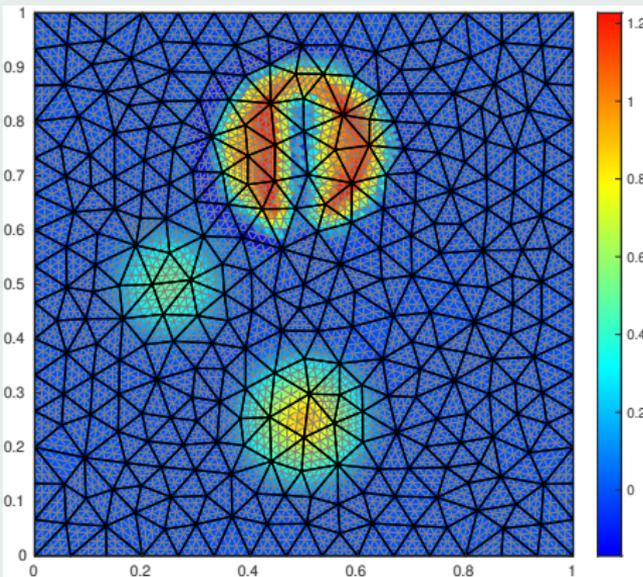
(a) Cartesian subdivision



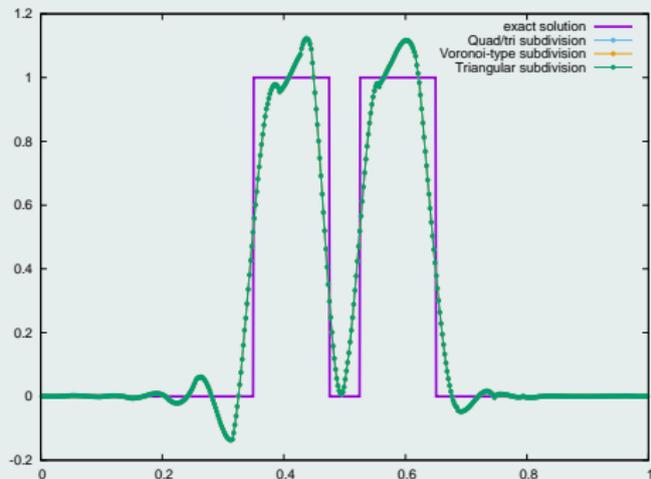
(b) Polygonal subdivision

Figure: \mathbb{P}^3 reconstructed flux FV schemes on 576 cells

Rotation of a composite signal after one full rotation



(c) Triangular subdivision



(d) Solution profiles for $y = 0.75$

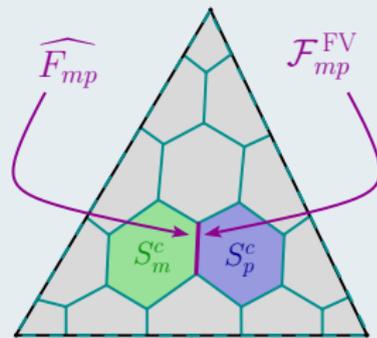
Figure: \mathbb{P}^3 reconstructed flux FV schemes on 576 cells

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Blending low and high order fluxes

- Each face f_{mp}^c of each subcell S_m^c will be assigned two fluxes

$$\widetilde{F}_{mp} = \mathcal{F}_{mp}^{FV} + \underbrace{\theta_{mp}}_{\in [0,1]} \underbrace{(\widehat{F}_{mp} - \mathcal{F}_{mp}^{FV})}_{\Delta F_{mp}}$$



- $\mathcal{F}_{mp}^{FV} := \mathcal{F}(\bar{u}_m^c, \bar{u}_p^v, \mathbf{n}_{mp})$

first-order subcell numerical flux

- \widehat{F}_{mp}

high-order DG reconstructed flux

- The local subcell monolithic DG/FV then writes as follows**

$$\frac{d\bar{u}_m^c}{dt} = -\frac{1}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp} \widetilde{F}_{mp}$$

Numerical flux

E-flux

$$\mathcal{F}(u^-, u^+, \mathbf{n}) = \frac{(\mathbf{F}(u^-) + \mathbf{F}(u^+)) \cdot \mathbf{n}}{2} - \frac{\gamma(u^-, u^+, \mathbf{n})}{2} (u^+ - u^-)$$

- $\gamma(u^-, u^+, \mathbf{n}) \geq \max_{w \in I(u^-, u^+)} (|\mathbf{F}'(w) \cdot \mathbf{n}|)$
- $I(a, b) = [\min(a, b), \max(a, b)]$
- **In the system case, we make use of either Rusanov, HLL, HLL-C, ...**

Time integration

- For sake of simplicity, we focus on forward Euler (FE) time stepping, as SSP Runge-Kutta can be formulated as convex combinations of FE
- The semi-discrete scheme provided with FE time integration writes

$$\bar{u}_m^{c, n+1} = \bar{u}_m^{c, n} - \frac{\Delta t}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp} \widetilde{F}_{mp}$$

Reformulation of the monolithic subcell scheme

- $\gamma_{mp} := \gamma(\bar{u}_m^{c,n}, \bar{u}_p^{v,n}, \mathbf{n}_{mp})$ 1st-order FV dissipation coefficient

$$\begin{aligned} \bullet \quad \bar{u}_m^{c,n+1} &= \bar{u}_m^{c,n} - \frac{\Delta t}{|S_m^c|} \left(\sum_{S_p^v \in \mathcal{V}_m^c} l_{mp} \widetilde{F}_{mp} \pm \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp} \gamma_{mp} \bar{u}_m^{c,n} + \mathbf{F}(\bar{u}_m^{c,n}) \cdot \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp} \mathbf{n}_{mp} \right) \\ &= \left(1 - \frac{\Delta t}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp} \gamma_{mp} \right) \bar{u}_m^{c,n} \\ &\quad + \frac{\Delta t}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp} \gamma_{mp} \left(\bar{u}_m^{c,n} - \frac{\widetilde{F}_{mp} - \mathbf{F}(\bar{u}_m^{c,n}) \cdot \mathbf{n}_{mp}}{\gamma_{mp}} \right) \end{aligned}$$

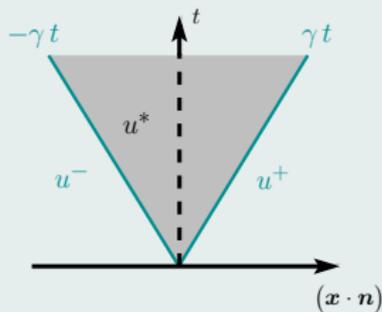
$$\bullet \quad \bar{u}_m^{c,n+1} = \left(1 - \frac{\Delta t}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp} \gamma_{mp} \right) \bar{u}_m^{c,n} + \frac{\Delta t}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp} \gamma_{mp} \widetilde{u}_{mp}^-$$

- **Convex combination under CFL condition**

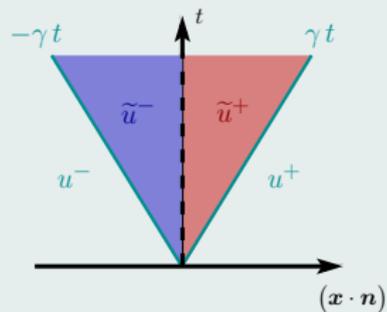
Blended Riemann intermediate states

$$\begin{aligned}
 \bullet \quad \widetilde{u}_{mp}^- &= \bar{u}_m^{c,n} - \frac{\mathcal{F}_{mp}^{FV} - \mathbf{F}(\bar{u}_m^{c,n}) \cdot \mathbf{n}_{mp}}{\gamma_{mp}} - \theta_{mp} \frac{\Delta F_{mp}}{\gamma_{mp}} \\
 &= \underbrace{\frac{\bar{u}_m^{c,n} + \bar{u}_p^{v,n}}{2} - \frac{(\mathbf{F}(\bar{u}_p^{v,n}) - \mathbf{F}(\bar{u}_m^{c,n})) \cdot \mathbf{n}_{mp}}{2 \gamma_{mp}}}_{u_{mp}^{*,FV}} - \theta_{mp} \frac{\Delta F_{mp}}{\gamma_{mp}}
 \end{aligned}$$

$$\bullet \quad \widetilde{u}_{mp}^\pm = u_{mp}^{*,FV} \pm \theta_{mp} \frac{\Delta F_{mp}}{\gamma_{mp}} \quad \Longrightarrow \quad u_{mp}^{*,FV} = \frac{1}{2} \left(\widetilde{u}_{mp}^- + \widetilde{u}_{mp}^+ \right)$$



(a) 1st-order situation



(b) Blended flux situation

Admissible solution

Find the correct θ_{mp}

- G a convex admissible set where the solution has to remain in

- $G = \{u_0 \in [\alpha, \beta] \implies u \in [\alpha, \beta]\}$

SCL

- $G = \left\{ u = \begin{pmatrix} \rho \\ \mathbf{q} \\ E \end{pmatrix}, \quad \rho > 0, p(u) > 0 \right\}$

Euler

- **A sufficient condition to ensure $\bar{u}_m^{c,n+1} \in G$ is that**

$$\forall S_p^v \in \mathcal{V}_m^c, \quad \widetilde{u}_{mp}^\pm = u_{mp}^{*,FV} \pm \theta_{mp} \frac{\Delta F_{mp}}{\gamma_{mp}} \in G$$

- We want to prevent to code from crashing (apparition of NaN, ...)
- We want to prevent the apparition of spurious oscillations
- *We want to ensure discrete entropy inequalities (?)*
- We apply a local blending coefficients smoothening to avoid too stiff transition from high to low orders

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Questions regarding entropy

- **Can we find the θ_{mp} coefficients ensuring an entropy inequality?**
- **What do we mean by entropy inequality?**
 - for one or any entropy?
 - at the discrete or semi-discrete time level?
 - at the cells or subcells space level?
- **If we manage to ensure an entropy inequality, is it worth the effort?**
 - in terms of accuracy
 - in terms of other critical properties to ensure, as positivity for instance
- **Do we really need an entropy inequality to practically capture the entropic weak solution?**

Definitions

- (η, ϕ) entropy - entropy flux
- $v(u) = \eta'(u)$ entropy variable
- $\psi(u) = v(u) \mathbf{F}(u) - \phi(u)$ entropy potential flux

Subcell entropy stability at the discrete level

for all (η, ϕ)

- if $\Delta F_{mp} \cdot (\bar{u}_p^{v,n} - \bar{u}_m^{c,n}) > 0$,

$$\theta_{mp} \leq \min \left(1, \frac{(\gamma_{mp} - \gamma_{\max}) (\bar{u}_p^{v,n} - \bar{u}_m^{c,n})}{2 \Delta F_{mp}} \right)$$

- $\gamma_{\max} := \max_{w \in I(\bar{u}_m^{c,n}, \bar{u}_p^{v,n})} (|\mathbf{F}'(w) \cdot \mathbf{n}_{mp}|)$

\implies

1st order scheme!

Semi-discrete subcell entropy dissipation

for a given (η, ϕ)

- if $\Delta F_{mp} \cdot \left(v(\bar{u}_p^{v,n}) - v(\bar{u}_m^{c,n}) \right) > 0$,

$$\theta_{mp} \leq \min \left(1, \frac{\left(\frac{\psi(\bar{u}_p^{v,n}) - \psi(\bar{u}_m^{c,n})}{v(\bar{u}_p^{v,n}) - v(\bar{u}_m^{c,n})} \right) \cdot \mathbf{n}_{mp} - \mathcal{F}_{mp}^{FV}}{\Delta F_{mp}} \right)$$

\implies 2nd order scheme!



A. RUEDA-RAMÍREZ, B. BOLM, D. KUZMIN AND G. GASSNER, *Monolithic Convex Limiting for Legendre-Gauss-Lobatto Discontinuous Galerkin Spectral Element Methods*. Commun. Appl. Math. Comput., 2024.

Sub-resolution basis functions

$$N_s = N_k$$

- Let $\{\varphi_m^c\}_m$ be the sub-resolution basis function such that, $\forall \psi \in \mathbb{P}^k(\omega_c)$

$$\int_{\omega_c} \varphi_m \psi \, dV = \int_{S_m^c} \psi \, dV$$

- Then, given $v_h^c \in \mathbb{P}^k(\omega_c)$, it writes

$$v_h^c = \sum_{m=1}^{N_k} \underline{v}_m^c \varphi_m^c$$

Semi-discrete cell entropy dissipation

for a given (η, ϕ)

- $\Delta \eta_c := \frac{d}{dt} \oint_{\omega_c} \eta(u_h^c) \, dV = \oint_{\omega_c} v(u_h^c) \partial_t u_h^c \, dV = \int_{\omega_c} v_h^c \partial_t u_h^c \, dV$

- $v_h^c = \sum_{m=1}^{N_k} \underline{v}_m^c \varphi_m^c$ L^2 projection of $v(u_h^c)$ onto \mathbb{P}^k

- $\Delta \eta_c = \sum_{m=1}^{N_k} \underline{v}_m^c \int_{\omega_c} \varphi_m^c \partial_t u_h^c \, dV = \sum_{m=1}^{N_k} |S_m^c| \underline{v}_m^c \frac{d \bar{u}_m^c}{dt} = - \sum_{m=1}^{N_k} \underline{v}_m^c \sum_{S_p^c \in \mathcal{V}_m^c} l_{mp} \widetilde{F}_{mp}$

Continuous Knapsack problem

- The sufficient condition rewrites as

$$\mathbf{C}_c \cdot \Theta_c \leq D_c$$

- Θ contains all the subcells' faces, \mathbf{C}_c the vector defined as

$$\mathbf{C}_{mp}^c = \begin{cases} \oint_{f_{mp}} \left((v(u_h^c) - \underline{v}_m^c) \mathcal{F}_n - (\psi(u_h^c) - \psi(u(\underline{v}_m^c))) \cdot \mathbf{n}_{mp} \right) dS, & \forall f_{mp} \subset \partial\omega_c \\ I_{mp} (\underline{v}_p^c - \underline{v}_m^c) \Delta F_{mp}, & \text{otherwise} \end{cases}$$

$$D_c = \sum_{f_{mp} \in \hat{f}_c} I_{mp} \psi(u(\underline{v}_m^c)) \cdot \mathbf{n}_{mp} - \sum_{f_{mp} \in \check{f}_c} I_{mp} (\underline{v}_p^c - \underline{v}_m^c) \mathcal{F}_{mp}^{FV}$$

$$\mathcal{F}_{mp}^{FV} = \mathcal{F}(u(\underline{v}_m^c), u(\underline{v}_p^v), \mathbf{n}_{mp}) \quad \text{modified FV numerical flux}$$

- Because** $D_c \geq 0$ **the Knapsack problem is indeed solvable**



Y. LIN AND J. CHAN, *High order entropy stable discontinuous Galerkin spectral element methods through subcell limiting*. JCP, 2024.

Greedy algorithm

- Find $\mathbf{0} \leq \Theta \leq \Theta^e \leq \mathbf{1}$ maximizing $\sum \theta_{mp}$ such that
- θ_{mp}^e being any given supplementary constraint

$$\mathbf{C}_c \cdot \Theta_c \leq D_c$$

High-order accuracy preservation

- Substituting u_h by a smooth solution u
- h_c is the diameter of cell ω_c
- Then, we have that

$$\mathbf{C}_c \cdot \mathbf{1} - D_c = |\omega_c| \mathcal{O}(h_c^{k+1})$$

- This finally implies that

$$\widetilde{F}_{mp} = \widehat{F}_{mp} + (\theta_{mp} - 1) \Delta F_{mp} = \widehat{F}_{mp} + \mathcal{O}(h_c^{k+2})$$

Linear advection of a sinus

$$\eta(u) = \frac{1}{2}u^2$$

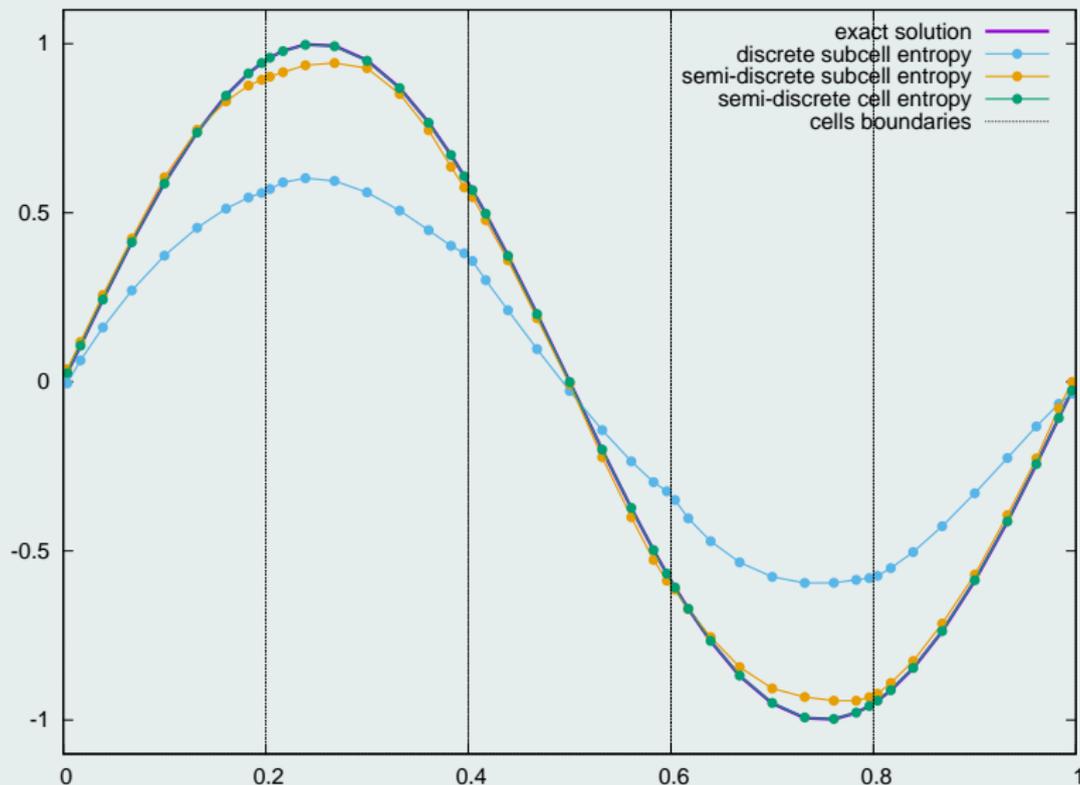


Figure: \mathbb{P}^8 -DG/FV with entropy stabilities on 5 cells

Linear advection of a composite signal

$$\eta(u) = \frac{1}{2}u^2$$

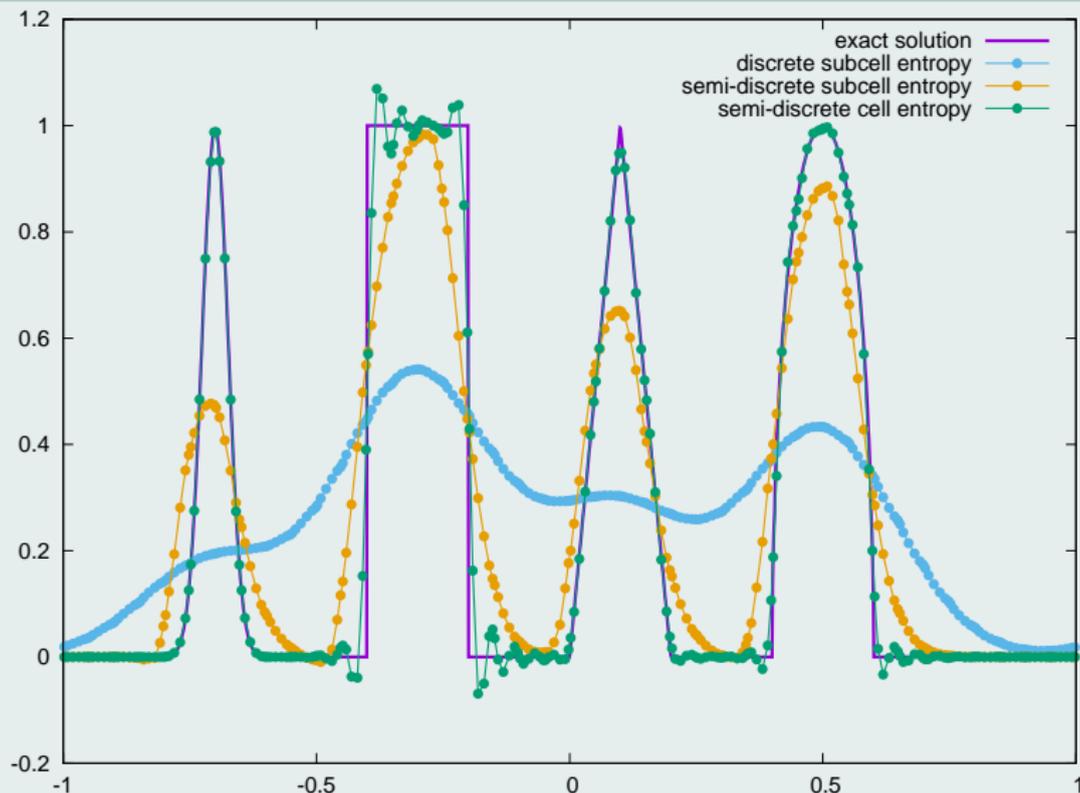


Figure: \mathbb{P}^5 -DG/FV with entropy stabilities on 40 cells

Linear advection of a composite signal $\eta(u) = |u - k_e|^{1+\epsilon} \setminus (1 + \epsilon)$

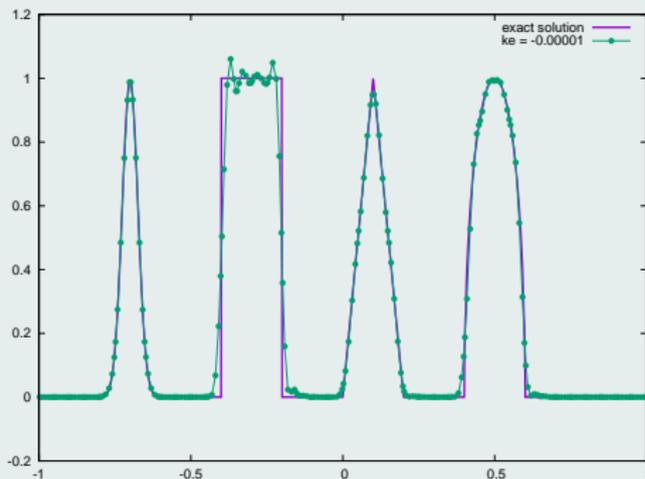
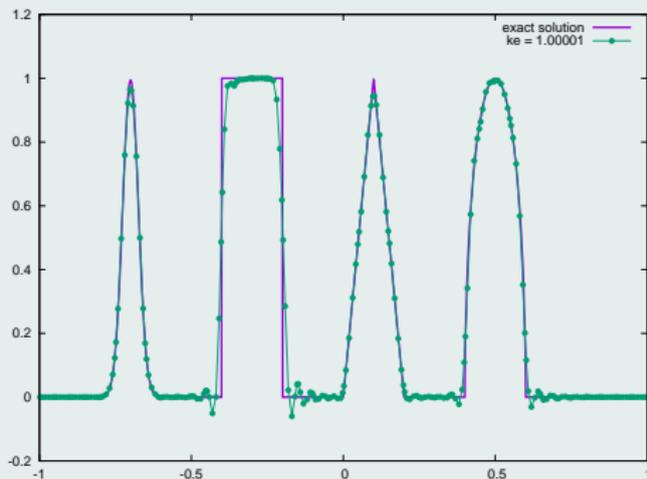
(a) $k_e = -0.00001$ (b) $k_e = 1.00001$

Figure: \mathbb{P}^5 -DG/FV submean values on 40 cells: $\epsilon = 0.25$

Linear advection of a composite signal $\eta(u) = |u - k_e|^{1+\epsilon} \setminus (1 + \epsilon)$

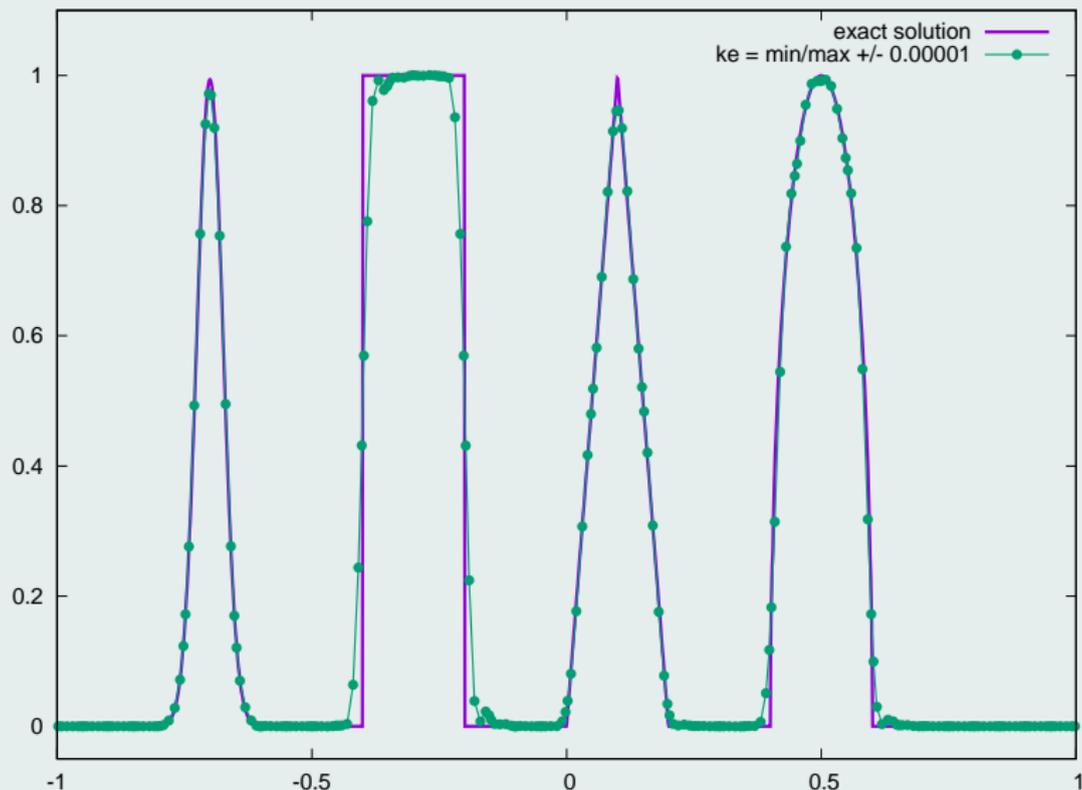


Figure: \mathbb{P}^5 -DG/FV on 40 cells: $k_e = -0.00001$ and $k_e = 1.00001$

Non-linear non-convex flux Buckley case

80 cells

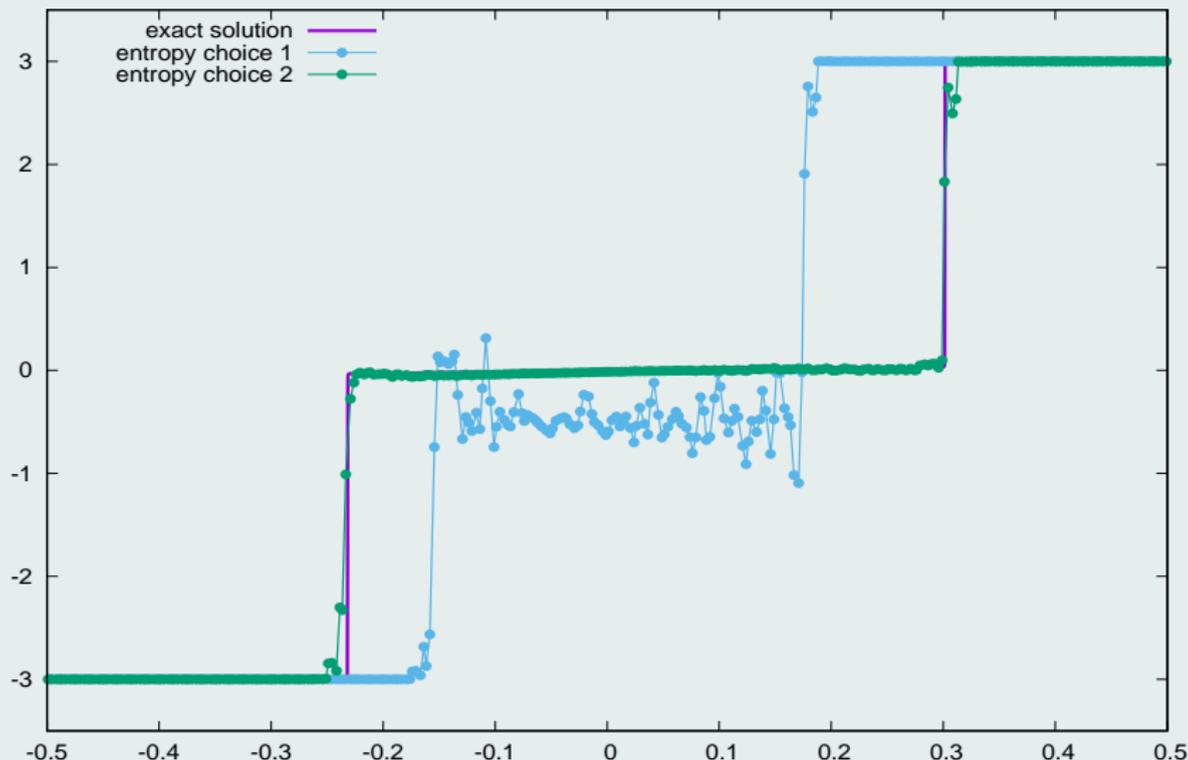


Figure: \mathbb{P}^3 -DG/FV submean values: $\eta_1(u) = \frac{1}{2}u^2$ and $\eta_2(u) = \int \text{atan}(20u) du$

Non-linear non-convex flux Buckley case

80 cells

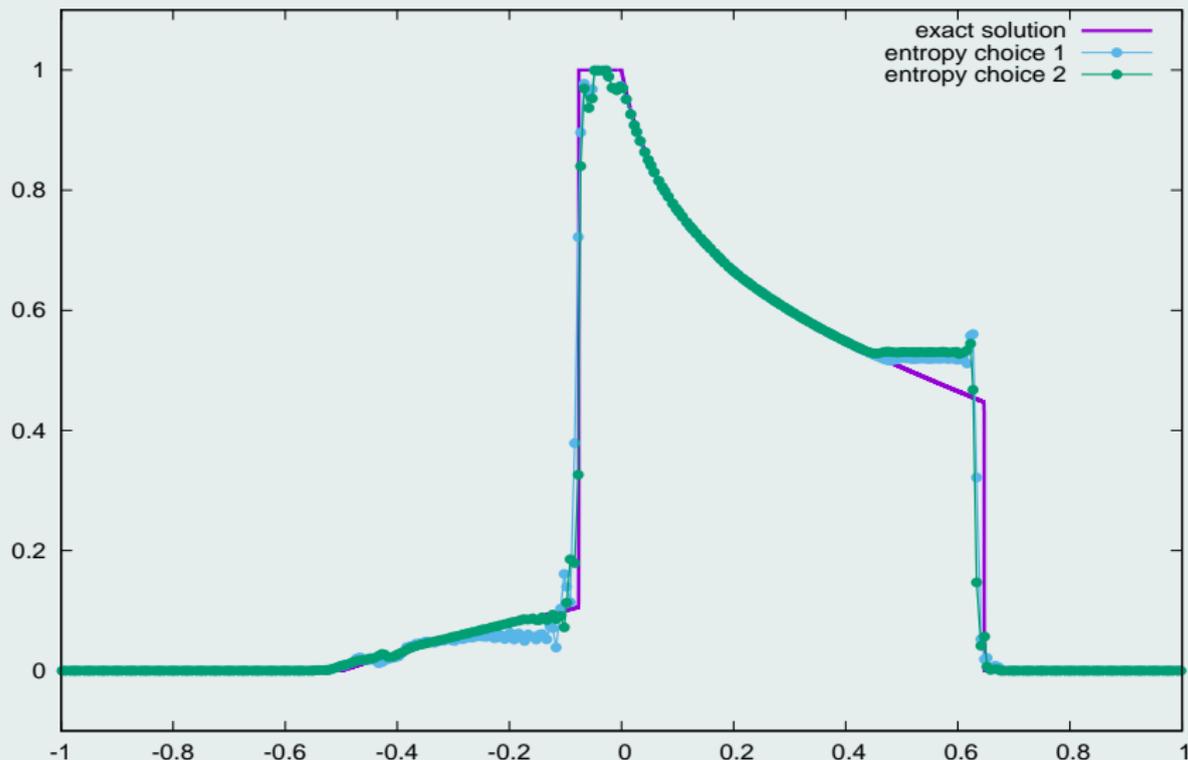


Figure: \mathbb{P}^3 -DG/FV submean values: $\eta_1(u) = \frac{1}{2}u^2$ and $\eta_2(u) = \int \text{atan}(20u) du$

KPP non-convex flux problem

$$\eta(u) = \frac{1}{2}u^2$$

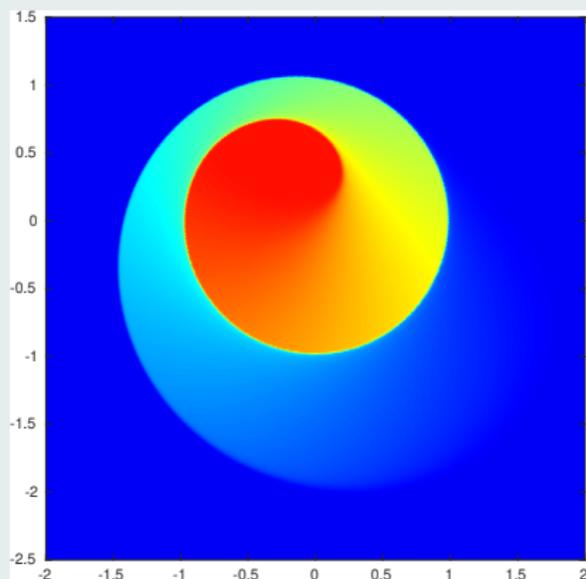
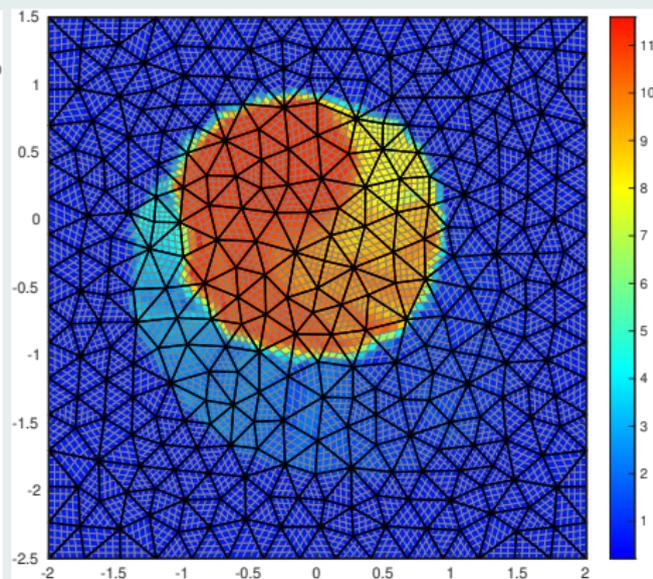
(a) 1th-order FV on 209184 cells(b) \mathbb{P}^4 -DG/FV on 576 cells

Figure: \mathbb{P}^4 -DG/FV entropic scheme: non-entropic solution

1D modified Sod shock tube problem

$$\eta(\mathbf{U}) = -\rho \log(p/\rho^\gamma)$$

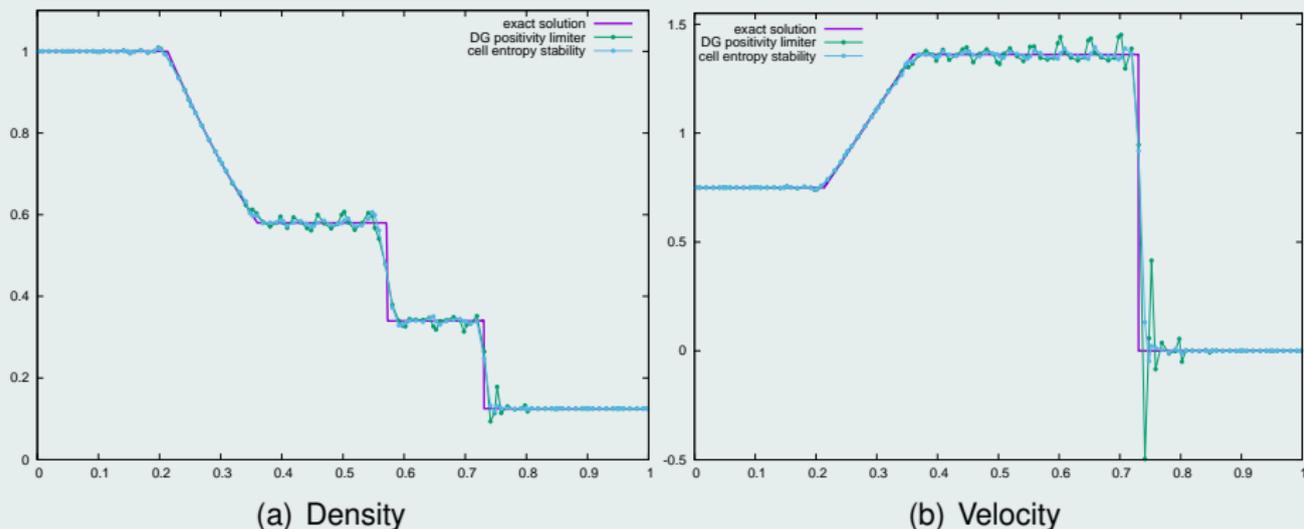


Figure: \mathbb{P}^5 pure DG with positivity limiter and \mathbb{P}^5 -DG/FV monolithic scheme with cell entropy stability on 20 cells

Questions regarding entropy \longrightarrow pieces of answer

- Can we find θ_{mp} coefficients ensuring an entropy inequality?

\hookrightarrow Yes!

- What do we mean by entropy inequality, and is it worth the effort?

- for any entropy, at the discrete time level and for any subcell

\hookrightarrow 1st-order

- for a given entropy, at the semi-discrete time level for any subcells

\hookrightarrow 2nd-order

- for a given entropy, at the semi-discrete time level for any cells

\hookrightarrow $(k + 1)$ th-order

\hookrightarrow $\mathcal{F}_{mp}^{FV} = \mathcal{F}(u(\underline{v}_m^c), u(\underline{v}_p^v), n_{mp})$



- Do we absolutely need entropy stability while aiming for high order?

\hookrightarrow Unclear... \implies **GMP and LMP + relaxation**

- 1 Introduction
- 2 DG as a subcell FV
- 3 Monolithic subcell DG/FV scheme
- 4 Entropy stabilities
- 5 Maximum principles**
- 6 Conclusion?

Global maximum principle

$$\bar{u}_m^{c,n+1} \in [\alpha, \beta]$$

$$\theta_{mp} \leq \min \left(1, \left| \frac{\gamma_{mp}}{\Delta F_{mp}} \right| \min (\beta - u_{mp}^{*,FV}, u_{mp}^{*,FV} - \alpha) \right)$$

Local maximum principle

$$\bar{u}_m^{c,n+1} \in [\alpha_m^c, \beta_m^c]$$

$$\bullet \alpha_m^c := \min_{S_q^w \in \mathcal{N}(S_m^c)} (\bar{u}_q^{w,n}) \quad \text{and} \quad \beta_m^c := \max_{S_q^w \in \mathcal{N}(S_m^c)} (\bar{u}_q^{w,n})$$

$$\theta_{mp} \leq \min \left(1, \left| \frac{\gamma_{mp}}{\Delta F_{mp}} \right| \begin{cases} \min (\beta_p^v - u_{mp}^{*,FV}, u_{mp}^{*,FV} - \alpha_m^c) & \text{if } \Delta F_{mp} > 0 \\ \min (\beta_m^c - u_{mp}^{*,FV}, u_{mp}^{*,FV} - \alpha_p^v) & \text{if } \Delta F_{mp} < 0 \end{cases} \right)$$

- The wider set $\mathcal{N}(S_m^c)$ is, the softer this local maximum principle will be
- **Smooth extrema relaxation to preserve accuracy**

Burgers equation

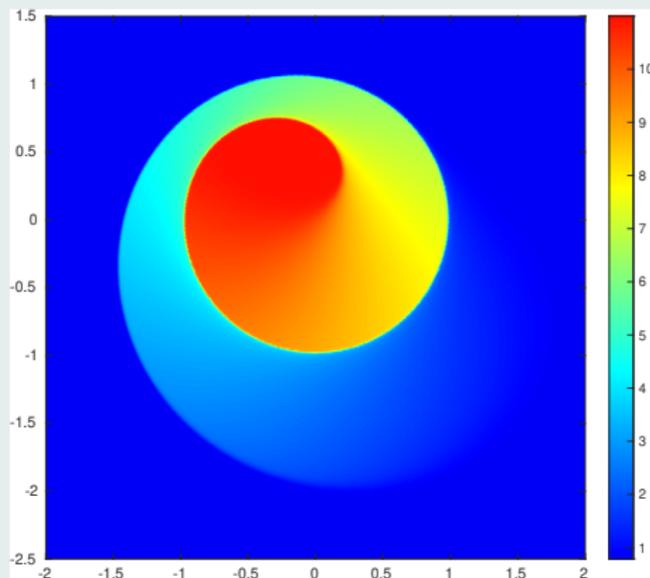
$$u_0(x, y) = \sin(2\pi(x + y))$$

(a) Solution submean values

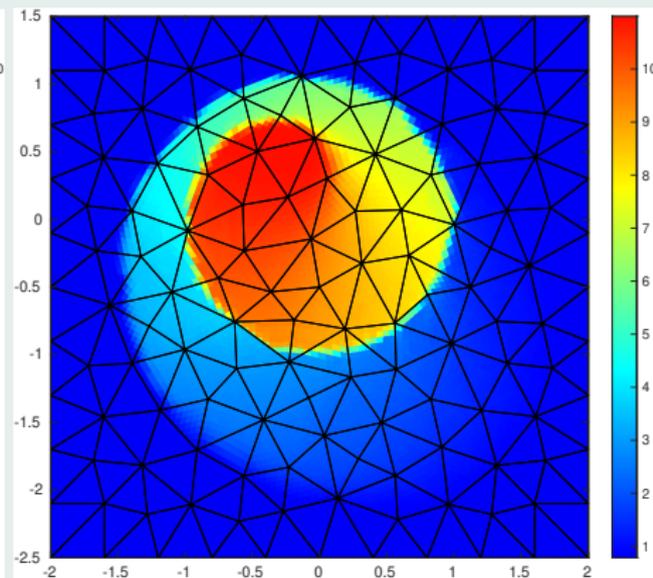
(b) Blending coefficients

Figure: \mathbb{P}^5 -DG/FV scheme with GMP and relaxed-LMP on 242 cells

KPP non-convex flux problem



(a) 1th-order FV on 209184 cells



(b) \mathbb{P}^7 -DG/FV on 272 cells

Figure: \mathbb{P}^7 -DG/FV scheme with GMP and relaxed-LMP

Non-linear Euler compressible gas dynamics equations

- $\partial_t \mathbf{V} + \nabla_x \cdot \mathbf{F}(\mathbf{V}) = \mathbf{0}$

- $\mathbf{V} = \begin{pmatrix} \rho \\ \mathbf{q} \\ E \end{pmatrix}$

conservative variables

- $\mathbf{F}(\mathbf{V}) = \begin{pmatrix} \mathbf{q} \\ \frac{\mathbf{q} \otimes \mathbf{q}}{\rho} + p I_d \\ (E + p) \frac{\mathbf{q}}{\rho} \end{pmatrix}$

flux function

- $p := p(\mathbf{V}) = (\gamma - 1) \left(E - \frac{1}{2} \frac{\|\mathbf{q}\|^2}{\rho} \right)$

equation of state

Monolithic subcell DG/FV scheme property

- Positivity of the density and internal energy, at the subcell scale

Definitions

- $$\widetilde{\mathbf{F}}_{mp} := \mathcal{F}_{mp}^{\text{FV}} + \Theta_{mp} \underbrace{(\widetilde{\mathbf{F}}_{mp} - \mathcal{F}_{mp}^{\text{FV}})}_{\Delta \mathbf{F}_{mp}}$$
convex blended flux

- $$\Theta_{mp} = \begin{pmatrix} \theta_{mp}^{\rho} & \mathbf{0} & 0 \\ \mathbf{0} & \theta_{mp}^q & \mathbf{0} \\ 0 & \mathbf{0} & \theta_{mp}^E \end{pmatrix}$$

- $$\mathcal{F}_{mp}^{\text{FV}} := \mathcal{F}(\overline{\mathbf{v}}_m^{c,n}, \overline{\mathbf{v}}_p^{v,n}, \mathbf{n}_{mp})$$
Global L-F, Rusanov, HLL(C), ...

Positivity of the density

- $$\theta_{mp}^{\rho} = \theta_{mp}^{(1)} \theta_{mp}^{(2)}$$

$$\theta_{mp}^{(1)} \leq \min \left(1, \left| \frac{\gamma_{mp}}{\Delta \mathbf{F}_{mp}^{\rho}} \right| \rho_{mp}^{*, \text{FV}} \right)$$

Positivity of the internal energy

- $A_{mp} = \frac{1}{(\gamma_{mp})^2} \left(\frac{1}{2} \|\Delta \mathbf{F}_{mp}^q\|^2 - \theta_{mp}^{(1)} \Delta F_{mp}^\rho \Delta F_{mp}^E \right)$
- $B_{mp} = \frac{1}{\gamma_{mp}} \left(\mathbf{q}_{mp}^{*,FV} \cdot \Delta \mathbf{F}_{mp}^q - \rho_{mp}^{*,FV} \Delta F_{mp}^E - \theta_{mp}^{(1)} E_{mp}^{*,FV} \Delta F_{mp}^\rho \right)$
- $M_{mp} = \rho_{mp}^{*,FV} E_{mp}^{*,FV} - \frac{1}{2} \|\mathbf{q}_{mp}^{*,FV}\|^2$

$$\theta_{mp}^{(2)} \leq \min \left(1, \frac{M_{mp}}{|B_{mp}| + \max(0, A_{mp})} \right)$$

- $\theta_{mp}^\rho = \theta_{mp}^{\rho(1)} \theta_{mp}^{(2)}, \quad \theta_{mp}^{q^x} = \theta_{mp}^{(2)}, \quad \theta_{mp}^{q^y} = \theta_{mp}^{(2)}, \quad \theta_{mp}^E = \theta_{mp}^{(2)}$



A. RUEDA-RAMÍREZ, B. BOLM, D. KUZMIN AND G. GASSNER, *Monolithic Convex Limiting for Legendre-Gauss-Lobatto Discontinuous Galerkin Spectral Element Methods*. Commun. Appl. Math. Comput., 2024.

LMP

$$\bar{v}_m^{c,n+1} \in [\alpha_m^c, \beta_m^c]$$

- $v \in \{\rho, q^x, q^y, E\}$

conservative variable

- $\alpha_m^c := \min_{S_q^w \in \mathcal{N}(S_m^c)} (\bar{v}_q^{w,n}, v_{mq}^{*,FV})$ and $\beta_m^c := \max_{S_q^w \in \mathcal{N}(S_m^c)} (\bar{u}_q^{w,n}, v_{mq}^{*,FV})$

$$\theta_{mp} \leq \min \left(1, \left| \frac{\gamma_{mp}}{\Delta F_{mp}} \right| \begin{cases} \min (\beta_p^v - u_{mp}^{*,FV}, u_{mp}^{*,FV} - \alpha_m^c) & \text{if } \Delta F_{mp} > 0 \\ \min (\beta_m^c - u_{mp}^{*,FV}, u_{mp}^{*,FV} - \alpha_p^v) & \text{if } \Delta F_{mp} < 0 \end{cases} \right)$$

- Smooth extrema relaxation to preserve accuracy**

Sod shock tube test case

10 cells

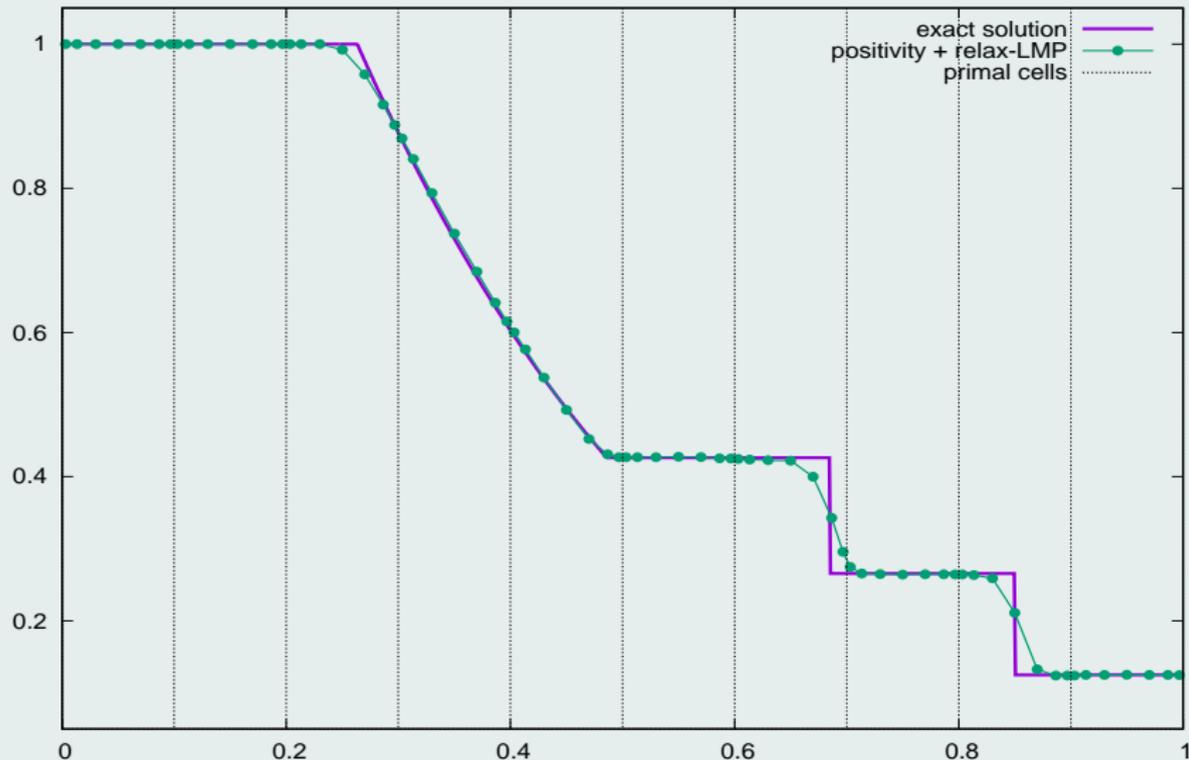
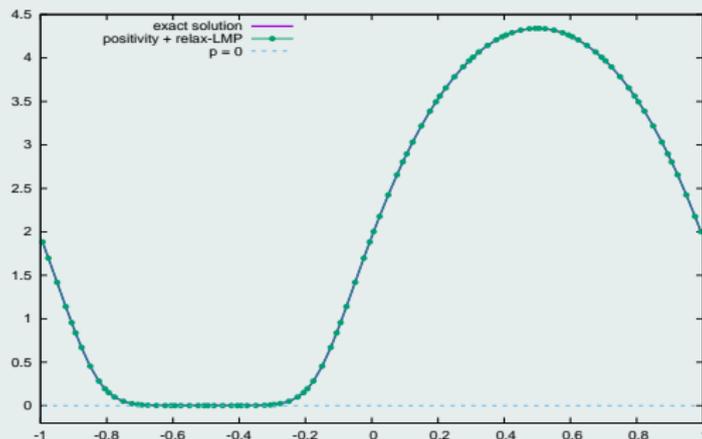


Figure: \mathbb{P}^6 -DG/FV scheme with GMP and relaxed-LMP: submean values

Smooth isentropic solution

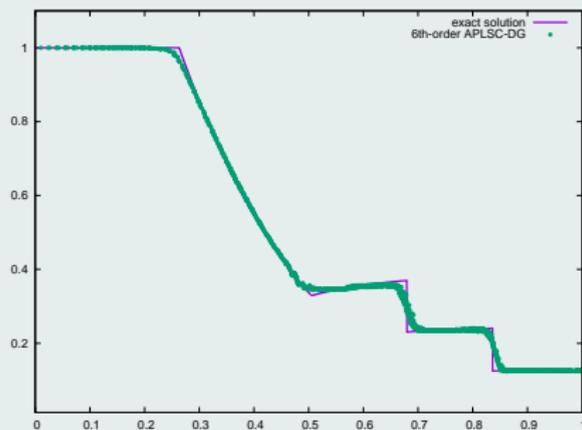
$$\rho_0 = 1 + 0.9999999 \sin(2\pi x)$$



h	L_1		L_2		θ_{mp}	
	$E_{L_1}^h$	$q_{L_1}^h$	$E_{L_2}^h$	$q_{L_2}^h$	min. θ_{mp}	aver. θ_{mp}
$\frac{1}{10}$	9.07E-4	5.86	1.23E-3	5.90	2.00E-1	0.981
$\frac{1}{20}$	1.56E-5	4.03	2.05E-5	3.83	1.92E-1	0.997
$\frac{1}{40}$	9.53E-7	4.89	1.44E-6	4.85	5.65E-4	0.999
$\frac{1}{80}$	3.21E-8	4.80	5.00E-8	4.87	3.48E-5	0.999
$\frac{1}{160}$	1.15E-9	-	1.71E-9	-	1.00	1.00

Table: Convergence rates of the \mathbb{P}^4 -DG/FV scheme with positivity and relaxed-LMP

Sod shock tube problem in cylindrical geometry

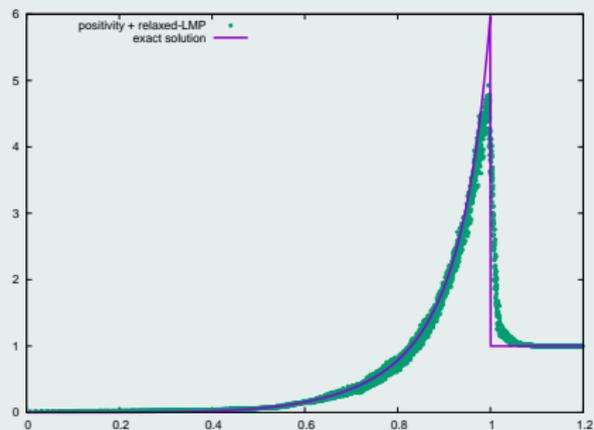


(b) Density profile

(a) Density map

Figure: \mathbb{P}^5 -DG/FV with positivity and relaxed-LMP on a 110 cells mesh

Sedov point blast problem in cylindrical geometry

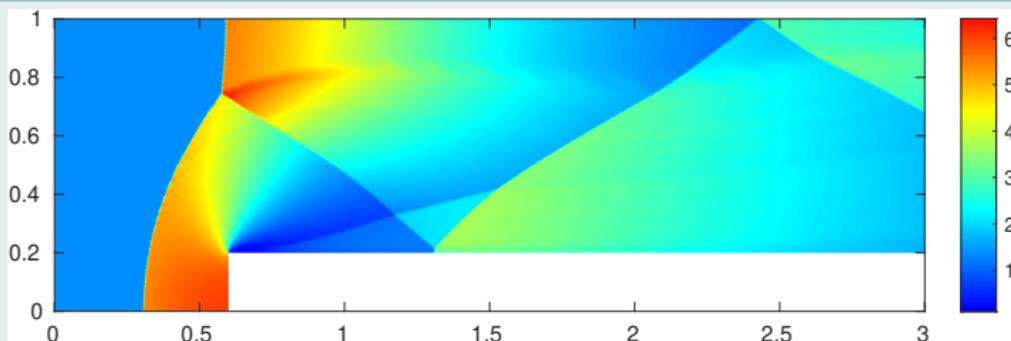


(b) Density profile

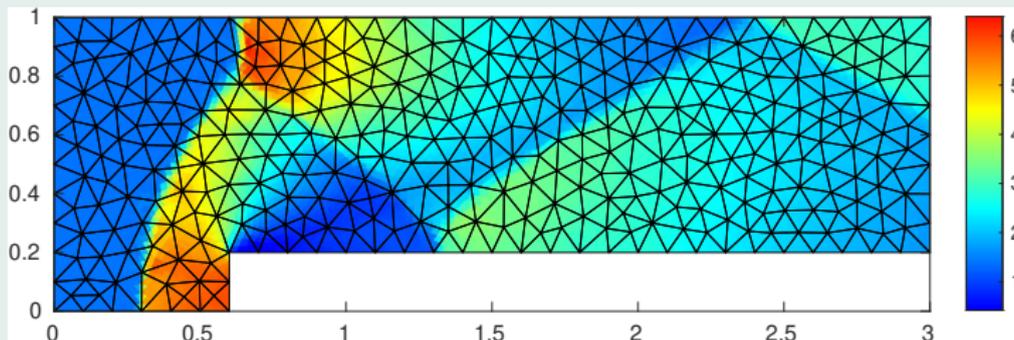
(a) Energy map

Figure: \mathbb{P}^5 -DG/FV with positivity and relaxed-LMP on a 271 cells mesh

Mach 3 forward-facing step



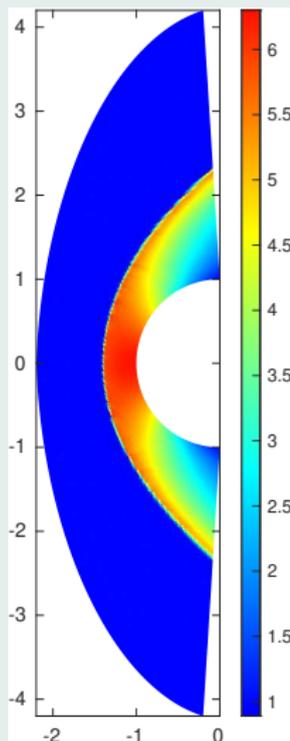
(a) \mathbb{P}^1 on 84108 cells



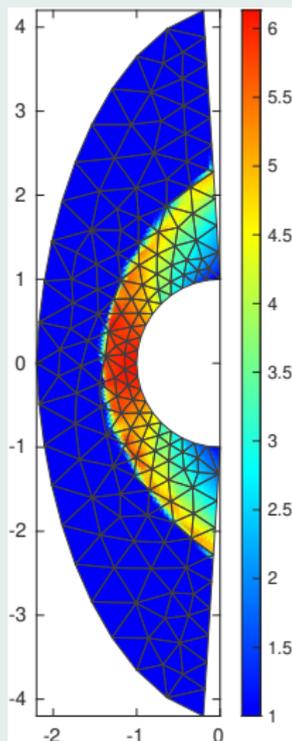
(b) \mathbb{P}^3 on 680 cells

Figure: Monolithic subcell DG/FV scheme with positivity and relaxed-LMP

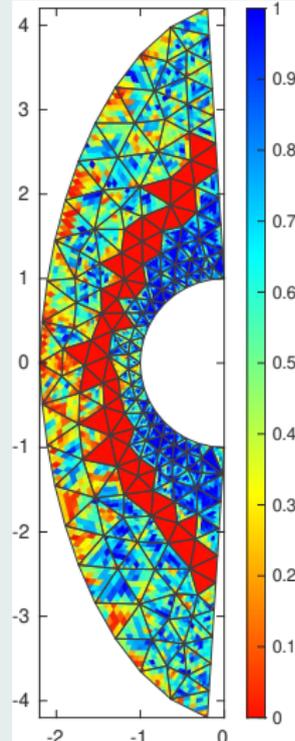
Mach 20 hypersonic flow over half cylinder



(a) \mathbb{P}^0 on 25266 cells



(b) \mathbb{P}^5 on 292 cells



(c) \mathbb{P}^5 on 292 cells

Figure: Monolithic subcell DG/FV scheme: density and blending coefficients

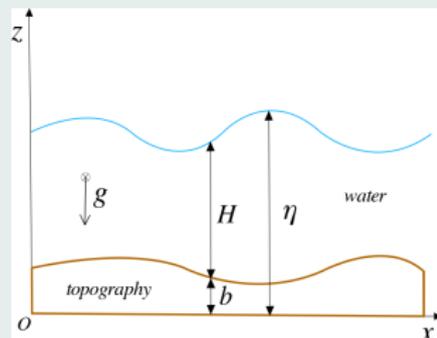
2D non-linear shallow water equations - prebalanced formulation

- $\partial_t \mathbf{V} + \nabla_x \cdot \mathbf{F}(\mathbf{V}, b) = \mathbf{B}(\mathbf{V}, \nabla_x b)$

- $\mathbf{V} = \begin{pmatrix} \eta \\ \mathbf{q} \end{pmatrix}$ conservative variables

- $\mathbf{B}(\mathbf{V}, \partial_x b) = \begin{pmatrix} 0 \\ -g\eta \nabla_x b \end{pmatrix}$ source term

- $\mathbf{F}(\mathbf{V}, b) = \begin{pmatrix} \mathbf{q} \\ \frac{\mathbf{q} \otimes \mathbf{q}}{\eta - b} + \frac{1}{2}g(\eta^2 - 2\eta b) I_d \end{pmatrix}$ flux function



Monolithic subcell DG/FV scheme property

- Positivity-preservation of the water height $H = \eta - b$, at the subcell scale
- Well-balancing property, at the subcell scale

Well-balancing property

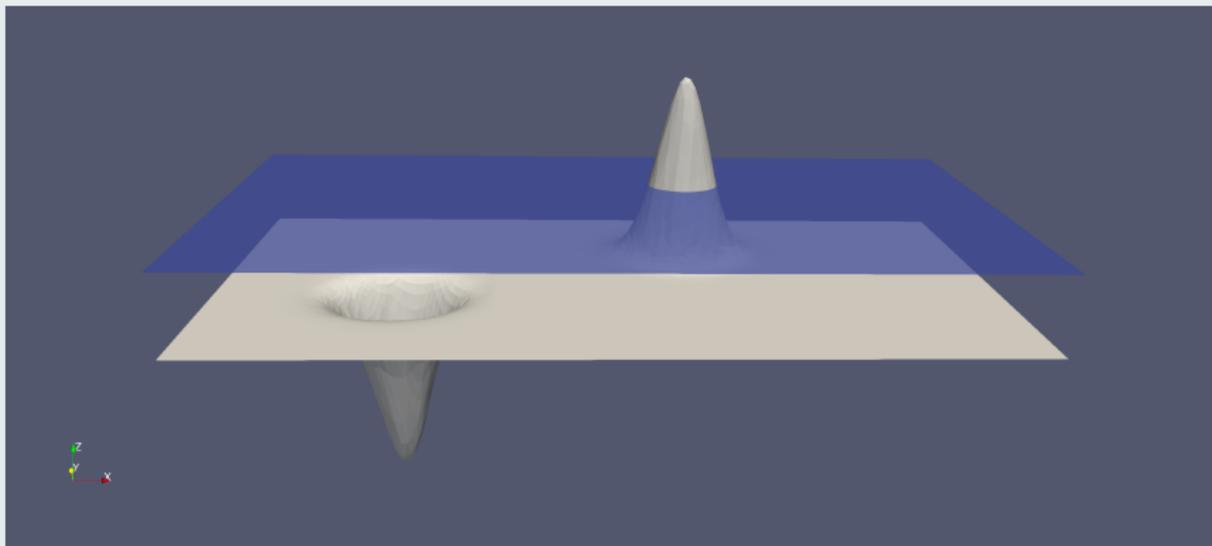
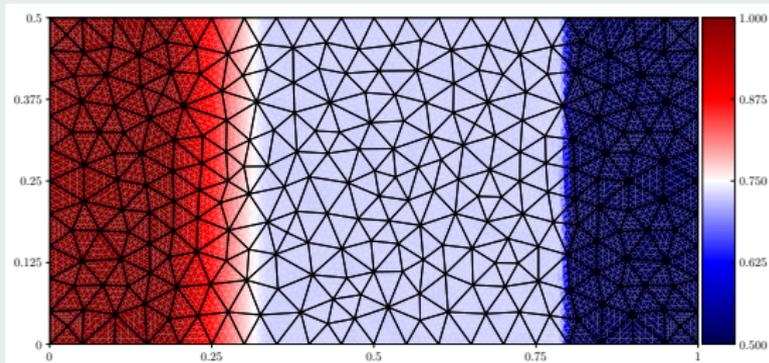
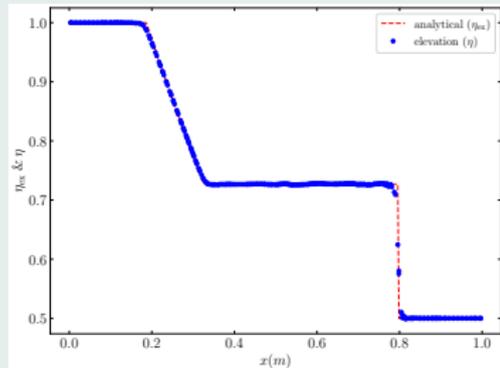


Figure : \mathbb{P}^2 -DG/FV with positivity and relaxed-LMP on a 2064 cells mesh

Dam break problem



(a) Elevation map



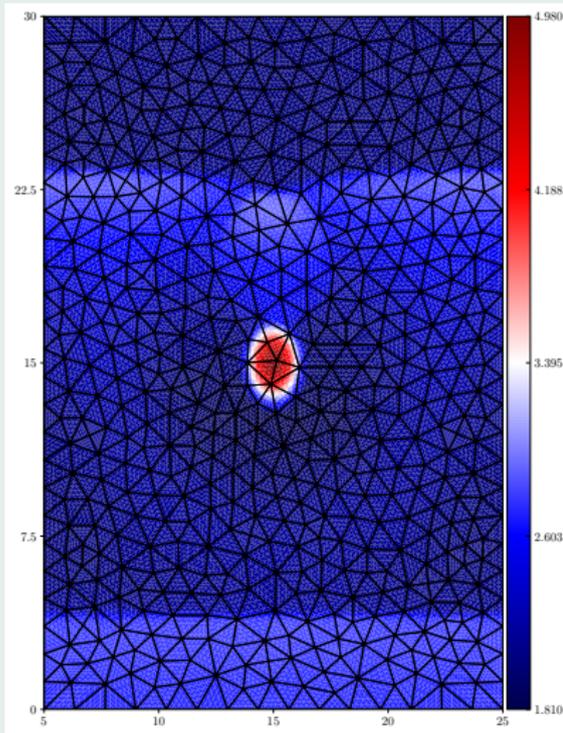
(b) Elevation profile

Figure : \mathbb{P}^5 -DG/FV with positivity and relaxed-LMP on a 506 cells mesh

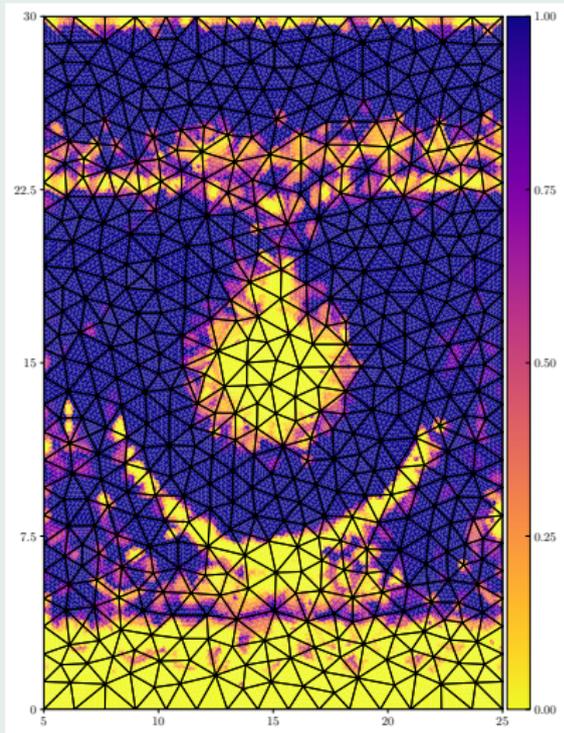
Rock-wave interaction

Figure : \mathbb{P}^6 -DG/FV with positivity and relaxed-LMP on a 1018 cells mesh

Rock-wave interaction



(a) Surface elevation



(b) Blending coefficients

Figure : \mathbb{P}^6 -DG/FV with positivity and relaxed-LMP on a 1018 cells mesh

- 1 Introduction
- 2 DG as a subcell FV
- 3 Monolithic subcell DG/FV scheme
- 4 Entropy stabilities
- 5 Maximum principles
- 6 Conclusion?**

Monolithic local subcell DG/FV scheme

- Reformulate DG schemes as subgrid FV-like schemes:
 - regardless the type of mesh used
 - regardless the space dimension (*in theory...*)
 - regardless the cell subdivision ($N_s \geq N_k$)
- Combine high-order reconstructed fluxes and 1st-order FV fluxes
 - ensuring a maximum or positivity preserving principle at the subcell scale
 - ensuring different entropy stability inequalities
 - reducing significantly the apparition of spurious oscillations
 - preserving the very accurate subcell resolution of DG schemes

Questions

- Is an entropy inequality for one entropy enough?

↪ **Generally, no**

- Is entropy inequality absolutely needed?

↪ **Maybe not** \implies **GMP + relaxed LMP**

Articles on this topic

-  **F.V**, *A Posteriori Correction of High-Order DG Scheme through Subcell Finite Volume Formulation and Flux Reconstruction*. JCP, 2018.
-  **A. HAIDAR, F. MARCHE AND F.V**, *A posteriori Finite-Volume local subcell correction of high-order discontinuous Galerkin schemes for the nonlinear shallow-water equations*. JCP, 2022.
-  **F.V AND R. ABGRALL**, *A posteriori local subcell correction of DG schemes through Finite Volume reformulation on unstructured grids*. SIAM SISC, 2023.
-  **A. HAIDAR, F. MARCHE AND F.V**, *Free-boundary problems for wave-structure interactions in shallow-water: DG-ALE description and local subcell correction*. JSC, 2023.
-  **A. HAIDAR, F. MARCHE AND F.V**, *A robust DG-ALE formulation for nonlinear shallow-water interactions with a floating object*. JSC, 2024.
-  **F.V**, *Monolithic local subcell DG/FV convex property preserving scheme on unstructured grids and entropy consideration*. JCP, 2024.

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-  **F.V**, *A Posteriori Correction of High-Order DG Scheme through Subcell Finite Volume Formulation and Flux Reconstruction*. JCP, 2018.
-  **A. HAIDAR, F. MARCHE AND F.V**, *A posteriori Finite-Volume local subcell correction of high-order discontinuous Galerkin schemes for the nonlinear shallow-water equations*. JCP, 2022.
-  **F.V AND R. ABGRALL**, *A posteriori local subcell correction of DG schemes through Finite Volume reformulation on unstructured grids*. SIAM SISC, 2023.
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-  **A. HAIDAR, F. MARCHE AND F.V**, *A robust DG-ALE formulation for nonlinear shallow-water interactions with a floating object*. JSC, 2024.
-  **F.V**, *Monolithic local subcell DG/FV convex property preserving scheme on unstructured grids and entropy consideration*. JCP, 2024.