Monolithic convex property preserving scheme

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Scalar Conservation Law (SCL)

- $\partial_t u(\mathbf{x},t) + \nabla_x \cdot \mathbf{F}(u(\mathbf{x},t)) = 0, \qquad (\mathbf{x},t) \in \mathbb{R}^d \times [0,T]$
- $u(\mathbf{x},0) = u_0(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d$

Fundamental difficulty

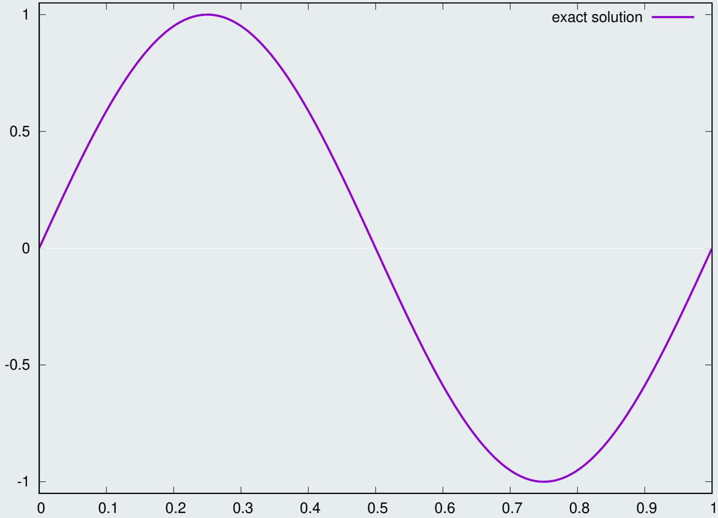
- **Even considering smooth flux function $F(\cdot)$ and initial datum $u_0(\cdot)$ ($\mathcal{C}^\infty$ for instance), solution may become discontinuous in finite time**
- Standard example: Burgers equation

$$\begin{cases} \partial_t u + \partial_x\left(\frac{1}{2}u^2\right) = 0, & (x,t) \in [0,1] \times \mathbb{R}^+, \\ u(x,0) = \sin(2\pi x), & x \in [0,1]\,. \end{cases}$$

Strong solution

- Local in time existence and uniqueness of a solution $u \in \mathcal{C}^1$

Formation of a discontinuity in finite time



Weak solution

- $\forall\, \psi \in \mathcal{C}_0^1(\mathbb{R}^d \times \mathbb{R}^+)$,

$$\int\int_{\mathbb{R}^d \times \mathbb{R}^+} \Big(u\, \partial_t \psi + \boldsymbol{F}(u) \,.\, \nabla_x \psi \Big) \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t - \int_{\mathbb{R}^d} u_0(\boldsymbol{x})\, \psi(\boldsymbol{x}, 0)\, \mathrm{d}\boldsymbol{x} = 0$$

- Non-uniqueness of a weak solution $u \in L^\infty_{loc}(\mathbb{R}^d \times \mathbb{R}^+)$

Entropy definition

- Let u be a strong solution
- η a strictly convex function is an entropy if $\exists\, \phi$ s.t. u also satisfies the following additional PDE

$$\partial_t\, \eta(u) + \nabla_x \,.\, \phi(u) = 0$$

- For SCL, any strictly convex function $\eta \in \mathcal{C}^2$ is an entropy, as with an entropy flux s.t $\phi'(u) = \eta'(u)\, \boldsymbol{F}'(u)$

$$\eta'(u)\, \big(\partial_t\, u + \boldsymbol{F}'(u) \,.\, \nabla_x u\big) = \partial_t\, \eta(u) + \nabla_x \,.\, \phi(u) = 0$$

Regularized problem and viscous solution $\varepsilon > 0$

- $\partial_t u^\varepsilon(\mathbf{x},t) + \nabla_x \,.\, \mathbf{F}(u^\varepsilon(\mathbf{x},t)) = \varepsilon\, \Delta_x u^\varepsilon(\mathbf{x},t), \qquad (\mathbf{x},t) \in \mathbb{R}^d \times [0,T]$
- $u^\varepsilon(\mathbf{x},0) = u_0^\varepsilon(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d$
- The smooth solution u^ε is assumed to converge to a function u as $\varepsilon \longrightarrow 0^+$
- Then, the function u
  - is a weak solution of the initial Cauchy problem
  - satisfies, for any entropy - entropy flux pair (η,ϕ) and $\forall\, \psi \in \mathcal{C}_0^1(\mathbb{R}^d \times \mathbb{R}^+)$,

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^+} \Big(\eta(u)\,\partial_t\psi + \phi(u)\,.\,\nabla_x\psi\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_{\mathbb{R}^d} \eta\big(u_0(\boldsymbol{x})\big)\,\psi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x} \geq 0$$

that we note for sake of simplicity as

$$\partial_t\,\eta(u) + \nabla_x \,.\, \phi(u) \leq 0 \qquad \textit{in a weak sense}$$

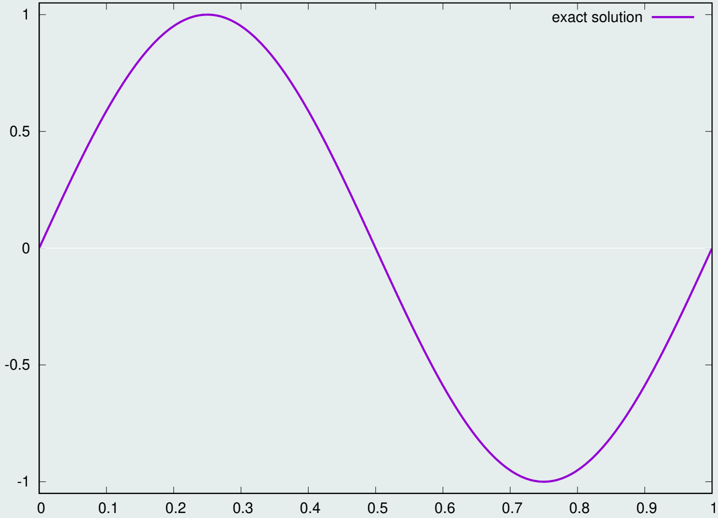
Proof *in a weak sense*

- $$\partial_t \eta(u^\varepsilon) + \nabla_x \cdot \phi(u^\varepsilon) = \varepsilon\, \eta'(u)\, \Delta_x u^\varepsilon$$
$$= \varepsilon\, \Delta_x \eta(u^\varepsilon) - \underbrace{\varepsilon\, \eta''(u^\varepsilon)\, \|\nabla_x u^\varepsilon\|^2}_{\geq 0}$$
$$\leq \varepsilon\, \Delta_x \eta(u^\varepsilon)$$
- Because $u^\varepsilon \xrightarrow[\varepsilon \longrightarrow 0^+]{} u$, the weak solution u satisfies
$$\partial_t \eta(u) + \nabla_x \cdot \phi(u) \leq 0$$

Unique entropic weak solution

- Considering $\boldsymbol{F} \in \mathcal{C}^2$ and $u_0 \in L^\infty(\mathbb{R}^d)$
- Cauchy problem admits a unique entropic weak solution $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+)$
- Furthermore, if $u_0 \in [\alpha, \beta] \implies u \in [\alpha, \beta]$ **Maximum principle**

Formation of a discontinuity in finite time



Challenges of numerical discretization

- **The weak solution we wish to approach may present areas of great regularity as well as discontinuities of great intensity**

$\longrightarrow$ **Smooth areas**: high accuracy

$\longrightarrow$ **Discontinuities**: strong robustness and stability

$\longrightarrow$ **Everywhere**: additional mathematical or physical constraints

$\hookrightarrow$ Maximum principle or positivity

$\hookrightarrow$ Entropy inequalities

$\hookrightarrow$ ...

Numerical schemes and discontinuous approximated solution

- **Finite Volume** **FV**
- (Weighted) Essentially Non-Oscillatory (W)-ENO
- **Discontinuous Galerkin** **DG**

Discontinuous Galerkin scheme

- Introduced by Reed and Hill in 1973 in the frame of the neutron transport
- Major development and improvements by B. Cockburn and C.-W. Shu in a series of seminal papers

Procedure

- Local variational formulation
- Piecewise polynomial approximation of the solution in the cells
- Choice of the numerical fluxes
- Time integration

Advantages

- Natural extension of Finite Volume method
- Excellent analytical properties (L_2 stability, $hp-$adaptivity, . . .)
- Extremely high accuracy (superconvergent for scalar conservation laws)
- Compact stencil (involve only face neighboring cells)

Scalar conservation law

- $\partial_t u(\mathbf{x},t) + \nabla_x \,.\, \mathbf{F}\left(u(\mathbf{x},t)\right) = 0, \qquad (\mathbf{x},t) \in \omega \times [0,T]$
- $u(\mathbf{x},0) = u_0(\mathbf{x}), \qquad \mathbf{x} \in \omega$

$(k+1)^{\text{th}}$ order semi-discretization

- $\{\omega_c\}_c$ a partition of ω, such that $\omega = \bigcup_c \omega_c$
- $u_h(\mathbf{x},t)$ the numerical solution, such that $u_{h|\omega_c} = u_h^c \in \mathbb{P}^k(\omega_c)$

$$u_h^c(\mathbf{x},t) = \sum_{m=1}^{N_k} u_m^c(t)\,\sigma_m^c(\mathbf{x})$$

- $\{\sigma_m^c\}_{m=1,\dots,N_k}$ a basis of $\mathbb{P}^k(\omega_c)$, with $N_k = \frac{(k+1)(k+2)}{2}$ in 2D.

Local variational formulation on ω_c

- $\displaystyle\int_{\omega_c} \frac{\partial u_h^c}{\partial t}\,\psi\,\mathrm{d}V = \int_{\omega_c} \mathbf{F}(u_h^c)\,.\,\nabla_x\psi\,\mathrm{d}V - \int_{\partial\omega_c} \psi\,\mathcal{F}_n\,\mathrm{d}S, \qquad \forall\,\psi \in \mathbb{P}^k(\omega_c)$
- $\mathcal{F}_n = \mathcal{F}\left(u_h^c, u_h^v, \mathbf{n}\right)$ **numerical flux**

Numerical example: solid body rotation

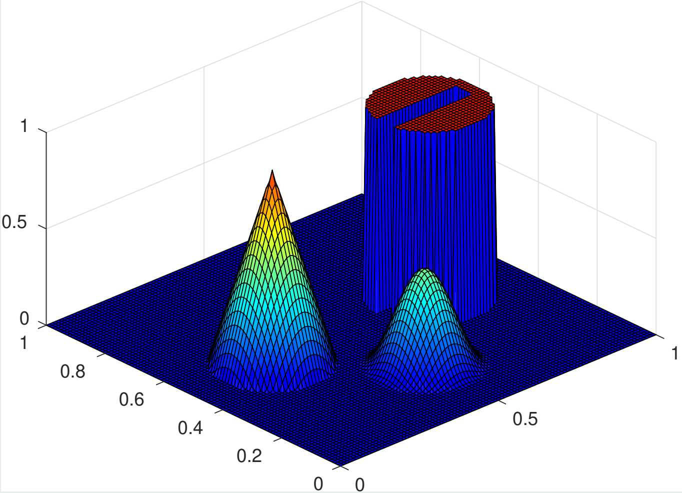
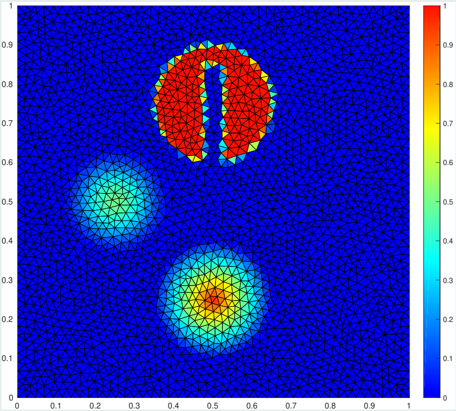
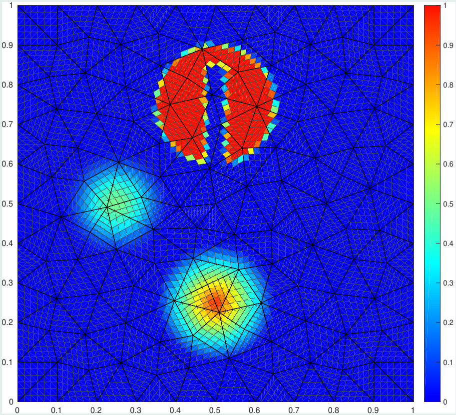


Figure: Rotation of composite signal: initial solution

Roughly constant number of degrees of freedom



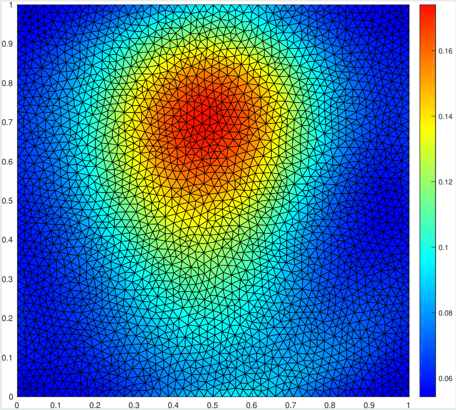
(a) 1st order on 5154 cells



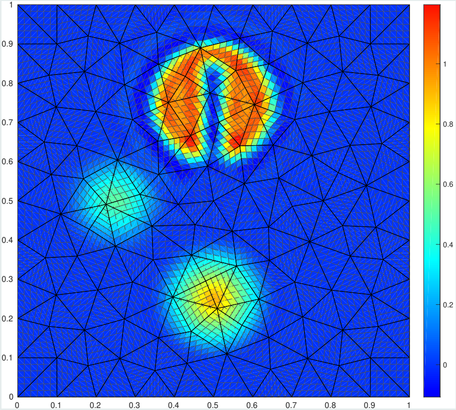
(b) 6th order on 242 cells (5082 DoF)

Figure: Rotation of composite signal: initial solution

Subcell resolution of DG scheme



(c) 1st order on 5154 cells



(d) 6th order on 242 cells (5082 DoF)

Figure: Rotation of composite signal after one period: subcells mean value

Subcell resolution of DG scheme $\sim$ **5 100 DoF**

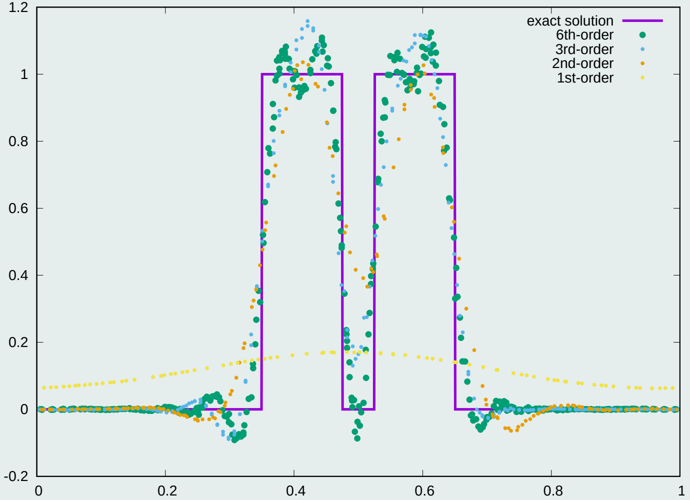
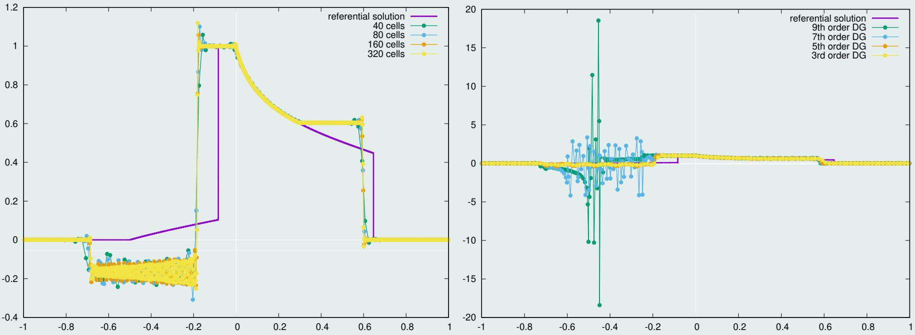


Figure: Rotation of composite signal after one period: profiles for $y = 0.75$

Buckley non-convex flux problem $F(u) = \frac{4\,u^2}{4\,u^2+(1-u)^2}$



(a) Non-entropic behavior (b) Aliasing phenomenon

Figure: Uncorrected DG solution for the Buckley non-convex flux case

Gibbs phenomenon and non-admissible solution

- High-order schemes leads to spurious oscillations near discontinuities
- Non-admissible solution potentially leading to a crash
- Vast literature of how prevent this phenomenon to happen:

$$\Longrightarrow \quad \textit{a priori} \text{ and } \textit{a posteriori} \text{ limitations}$$

A priori limitation

- Artificial viscosity
- Flux limitation
- Slope/moment limiter
- Hierarchical limiter
- ENO/WENO limiter

A posteriori limitation

- MOOD (“Multi-dimensional Optimal Order Detection”)
- Subcell finite volume limitation
- **A posteriori local subcell correction**

Admissible numerical solution

- Maximum principle / positivity preserving
- Limit the apparition of spurious oscillations
- Ensure a correct entropic behavior

Preserving high-accuracy and subcell resolution

Reduce the characteristic length of action

Methodology

Blend, at the subcell scale, high-order DG and 1st-order FV

F.V, *A Posteriori Correction of High-Order DG Scheme through Subcell Finite Volume Formulation and Flux Reconstruction.* JCP, 2018.

F.V AND **R. ABGRALL**, *A posteriori local subcell correction of DG through FV reformulation on unstructured grids.* SIAM SISC, 2023.

DG as a subcell Finite Volume

- Rewrite DG scheme as a FV-like scheme on a subgrid

Cell subdivision into $N_S \geq N_k$ subcells

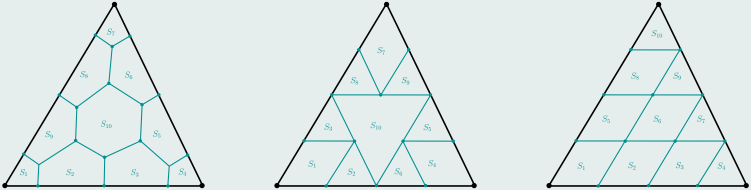


Figure: Examples of $N_s = N_k$ subdivision for $\mathbb{P}^3$ DG scheme on a triangle

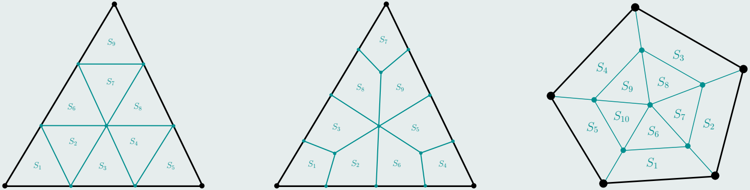


Figure: Examples of $N_S \geq N_k$ subdivision

DG schemes through residuals

- $$\sum_{m=1}^{N_k} \frac{\mathrm{d}\, u_m^c}{\mathrm{d}t} \int_{\omega_c} \sigma_m \, \sigma_p \, \mathrm{d}V = \int_{\omega_c} \mathbf{F}(u_h^c) \,\boldsymbol{.}\, \nabla_x \sigma_p \, \mathrm{d}V - \int_{\partial\omega_c} \sigma_p \, \mathcal{F}_n \, \mathrm{d}S, \quad \forall\, p \in [\![1, N_k]\!]$$

$$\Longrightarrow \quad \boxed{M_c \, \frac{\mathrm{d}\, \mathrm{U}_c}{\mathrm{d}t} = \Phi_c}$$

- $(\mathrm{U}_c)_m = u_m^c$ — Solution moments
- $(M_c)_{mp} = \displaystyle\int_{\omega_c} \sigma_m \, \sigma_p \, \mathrm{d}V$ — Mass matrix
- $(\Phi_c)_m = \displaystyle\int_{\omega_c} \mathbf{F}(u_h^c) \,\boldsymbol{.}\, \nabla_x \sigma_m \, \mathrm{d}V - \int_{\partial\omega_c} \sigma_m \, \mathcal{F}_n \, \mathrm{d}S$ — DG residuals

Subdivision and definition

- ω_c is subdivided into N_s subcells S_m^c
- Let us define $\overline{\psi}_m^c = \dfrac{1}{|S_m^c|} \displaystyle\int_{S_m^c} \psi \, \mathrm{d}V$ the subcell mean value

Submean values

- $\overline{u}_m^c = \dfrac{1}{|S_m^c|} \displaystyle\sum_{q=1}^{N_k} u_q^c \int_{S_m^c} \sigma_q \, \mathrm{d}V \quad \Longrightarrow \quad \boxed{\overline{\mathrm{U}}_c = P_c \, \mathrm{U}_c}$
- $(\overline{\mathrm{U}}_c)_m = \overline{u}_m^c$ Submean values
- $(P_c)_{mp} = \dfrac{1}{|S_m^c|} \displaystyle\int_{S_m^c} \sigma_p \, \mathrm{d}V$ Projection matrix

$$\Longrightarrow \quad \boxed{\frac{\mathrm{d}\,\overline{\mathrm{U}}_c}{\mathrm{d}t} = P_c \, M_c^{-1} \, \Phi_c}$$

Admissibility of the cell sub-partition into subcells

- $P_c^{\mathrm{t}} \, P_c$ **has to be non-singular**

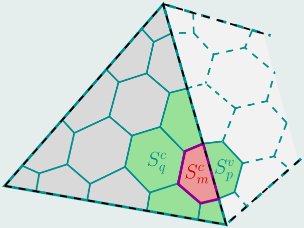
$$\Longrightarrow \quad \boxed{\mathrm{U}_c = (P_c^{\mathrm{t}} \, P_c)^{-1} P_c^{\mathrm{t}} \, \overline{\mathrm{U}}_c}$$ Least square procedure

- If $N_s = N_k, \quad \overline{\mathrm{U}}_c = P_c \, \mathrm{U}_c \iff \mathrm{U}_c = P_c^{-1} \overline{\mathrm{U}}_c$

Subcell Finite Volume: reconstructed fluxes

- Let us introduce the **reconstructed fluxes**

$$\boxed{\frac{\mathrm{d}\,\overline{u}_m^c}{\mathrm{d}t} = -\frac{1}{|S_m^c|}\sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\widehat{F_{pm}}}$$



- $\mathcal{V}_m^c$ the set of face neighboring subcells of S_m^c
- We impose that on the boundary of cell ω_c, so for $S_p^v \notin \omega_c$

$$\boxed{l_{mp}\,\widehat{F_{pm}} = \int_{f_{mp}^c} \mathcal{F}_n \,\mathrm{d}S \equiv \int_{f_{mp}^c} \mathcal{F}\left(u_h^c,\, u_h^v,\, \boldsymbol{n}_{mp}^c\right)\,\mathrm{d}S}$$

- $\breve{\mathcal{V}}_m^c$ the set of face neighboring subcells of S_m^c belonging to ω_c
- Let A_c be the adjacency matrix such that

$$(A_c)_{mp} = \begin{cases} 1 & \text{if } S_p^v \in \breve{\mathcal{V}}_m^c \text{ with } m < p, \\ -1 & \text{if } S_p^v \in \breve{\mathcal{V}}_m^c \text{ with } m > p, \\ 0 & \text{if } S_p^v \notin \breve{\mathcal{V}}_m^c. \end{cases}$$

Subcell Finite Volume: reconstructed fluxes

- Let us introduce $D_c = \text{diag}\left(|S_1^c|, \dots, |S_{N_k}^c|\right)$ and $(\mathrm{B}_c)_m = \int_{\partial S_m^c \cap \partial \omega_c} \mathcal{F}_n \, \mathrm{d}S$
- Let $\widehat{\mathrm{F}_c}$ be the vector containing all the interior faces reconstructed fluxes

$$\boxed{-A_c \, \widehat{\mathrm{F}_c} = D_c \, P_c \, M_c^{-1} \, \Phi_c + \mathrm{B}_c}$$

Graph Laplacian technique

- $A_c \in \mathcal{M}_{N_s \times N_f^c}$ with N_f^c the number of interior faces
- $(L_c)_{mp} := \left(A_c \, A_c^{\mathrm{t}}\right)_{mp} = \begin{cases} \left|\, \breve{\mathcal{V}}_m^c \,\right| & \text{if } m = p, \\ -1 & \text{if } S_p^v \in \breve{\mathcal{V}}_m^c, \\ 0 & \text{otherwise.} \end{cases}$
- $L_c \, \mathbf{1} = \mathbf{0}$ where $\mathbf{1} = (1, \dots, 1)^{\mathrm{t}} \in \mathbb{R}^{N_s}$
- $\Pi = \frac{1}{N_s} \, (\mathbf{1} \otimes \mathbf{1}) \in \mathcal{M}_{N_s}$

Graph Laplacian technique

- Let $\mathcal{L}_c^{-1}$ be the pseudo-inverse of L_c such that

$$\mathcal{L}_c^{-1} = (L_c + \lambda\,\Pi)^{-1} - \frac{1}{\lambda}\,\Pi \qquad \forall\,\lambda \neq 0$$

- Then, $\widehat{\mathrm{F}_c}$ is uniquely defined as following

$$\boxed{\widehat{\mathrm{F}_c} = -A_c^{\mathrm{t}}\,\mathcal{L}_c^{-1}\left(D_c\,P_c\,M_c^{-1}\,\Phi_c + \mathrm{B}_c\right)}$$

- **The only terms depending on the time are Φ_c and $\mathbf{B}_c$**
- Equivalently, the polynomial solution governing equation is given by

$$\frac{\mathrm{d}\,\mathrm{U}_c}{\mathrm{d}t} = -P_c^{-1}\,D_c^{-1}\left(A_c\,\widehat{\mathrm{F}_c} + \mathrm{B}_c\right)$$

remark

- This unique solution does exit since $\left(D_c\,P_c\,M_c^{-1}\,\Phi_c + \mathrm{B}_c\right) \mathbf{.}\,\mathbf{1} = 0$

Different cell subdivisions

$\mathbb{P}^3$

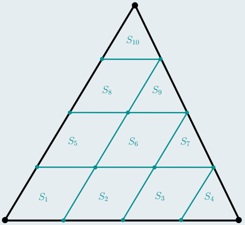
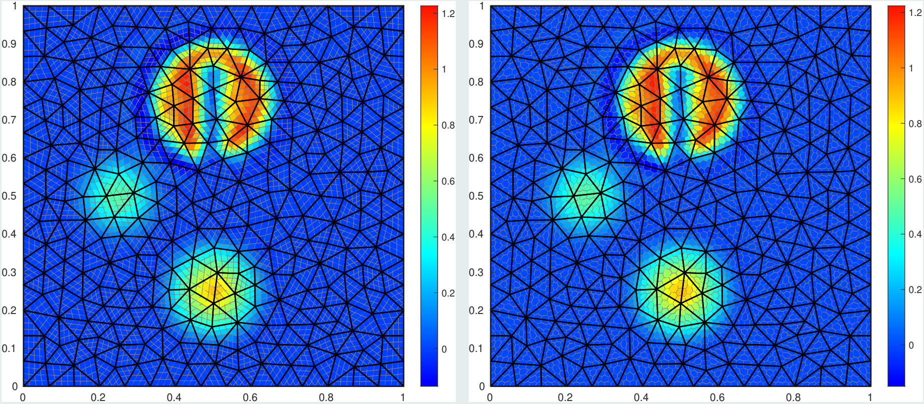


Figure: Examples of easily generalizable subdivisions for a triangle cell

DG is DG

- **Only the functional space matters**
- **The cell subdivision has no influence on the resulting scheme**
- **Even in the case where** $N_s > N_k$

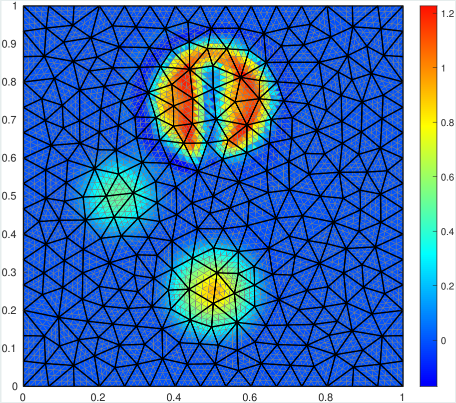
Rotation of a composite signal after one full rotation



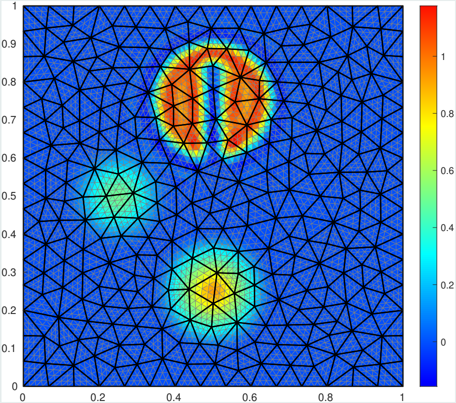
(a) Cartesian subdivision (b) Polygonal subdivision

Figure: $\mathbb{P}^3$ reconstructed flux FV schemes on 576 cells: subcells mean values

Rotation of a composite signal after one full rotation



(a) Triangular subdivision



(b) Enriched-DG triangular subdivision

Figure: $\mathbb{P}^3$ and $\mathbb{P}^{4+\frac{1}{6}}$ reconstructed flux FV schemes on 576 cells: subcells mean values

Rotation of a composite signal after one full rotation

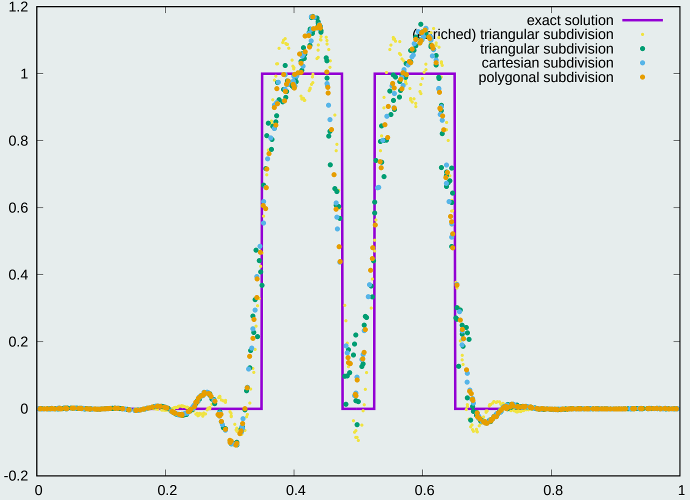
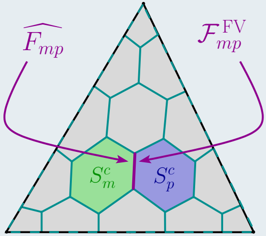


Figure: Reconstructed flux FV schemes on 576 cells: solution profiles for $y = 0.75$

Blending low and high order fluxes

- Each face f_{mp}^c of each subcell S_m^c will be assigned two fluxes

- $$\widetilde{F_{mp}} = \mathcal{F}_{mp}^{\mathrm{FV}} + \underbrace{\theta_{mp}}_{\in[0,1]} \underbrace{\left(\widehat{F_{mp}} - \mathcal{F}_{mp}^{\mathrm{FV}}\right)}_{\Delta F_{mp}}$$



- $\mathcal{F}_{mp}^{\mathrm{FV}} := \mathcal{F}\left(\overline{u}_m^c, \overline{u}_p^v, \boldsymbol{n}_{mp}\right)$ first-order subcell numerical flux
- $\widehat{F_{mp}}$ high-order DG reconstructed flux
- **The local subcell monolithic DG/FV then writes as follows**

$$\frac{\mathrm{d}\,\overline{u}_m^c}{\mathrm{d}t} = -\frac{1}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\, \widetilde{F_{mp}}$$

Numerical flux — E-flux

$$\mathcal{F}(u^-, u^+, \boldsymbol{n}) = \frac{(\boldsymbol{F}(u^-) + \boldsymbol{F}(u^+))}{2} \boldsymbol{.}\, \boldsymbol{n} - \frac{\gamma(u^-, u^+, \boldsymbol{n})}{2} (u^+ - u^-)$$

- $\gamma(u^-, u^+, \boldsymbol{n}) \geq \max\limits_{w \in \mathrm{I}(u^-, u^+)} (|\boldsymbol{F}'(w) \boldsymbol{.}\, \boldsymbol{n}|)$
- $\mathrm{I}(a, b) = [\min(a, b), \max(a, b)]$
- **In the system case, we make use of either Rusanov, HLL, HLL-C, . . .**

Time integration

- For sake of simplicity, we focus on forward Euler (FE) time stepping, as SSP Runge-Kutta can be formulated as convex combinations of FE
- The semi-discrete scheme provided with FE time integration writes

$$\overline{u}_m^{c,n+1} = \overline{u}_m^{c,n} - \frac{\Delta t}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\, \widetilde{F_{mp}}$$

Reformulation of the monolithic subcell scheme

- $\gamma_{mp} := \gamma\left(\overline{u}_m^{c,n}, \overline{u}_p^{v,n}, \boldsymbol{n}_{mp}\right)$ 1st-order FV dissipation coefficient

- $$\overline{u}_m^{c,n+1} = \overline{u}_m^{c,n} - \frac{\Delta t}{|S_m^c|}\left(\sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\widetilde{F_{mp}} \pm \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\gamma_{mp}\,\overline{u}_m^{c,n} + \boldsymbol{F}(\overline{u}_m^{c,n}) \cdot \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\boldsymbol{n}_{mp}\right)$$

$$= \left(1 - \frac{\Delta t}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\gamma_{mp}\right) \overline{u}_m^{c,n}$$

$$+ \frac{\Delta t}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\gamma_{mp}\left(\overline{u}_m^{c,n} - \frac{\widetilde{F_{mp}} - \boldsymbol{F}(\overline{u}_m^{c,n}) \cdot \boldsymbol{n}_{mp}}{\gamma_{mp}}\right)$$

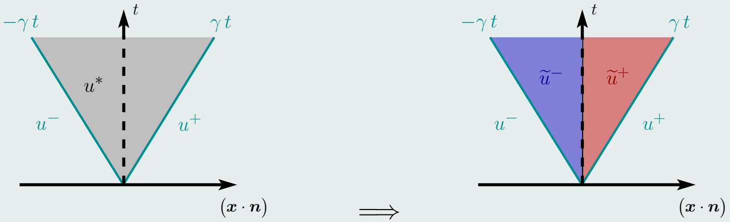
- $$\boxed{\overline{u}_m^{c,n+1} = \left(1 - \frac{\Delta t}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\gamma_{mp}\right) \overline{u}_m^{c,n} + \frac{\Delta t}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\gamma_{mp}\,\widetilde{u_{mp}^-}}$$

- **Convex combination under CFL condition**

Blended Riemann intermediate states

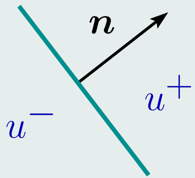
- $$\widetilde{u_{mp}}^{-} = \overline{u}_m^{c,n} - \frac{\mathcal{F}_{mp}^{\text{FV}} - \boldsymbol{F}(\overline{u}_m^{c,n})\cdot\boldsymbol{n}_{mp}}{\gamma_{mp}} - \theta_{mp}\frac{\Delta F_{mp}}{\gamma_{mp}}$$
$$= \underbrace{\frac{\overline{u}_m^{c,n}+\overline{u}_p^{v,n}}{2} - \frac{\left(\boldsymbol{F}(\overline{u}_p^{v,n}) - \boldsymbol{F}(\overline{u}_m^{c,n})\right)\cdot\boldsymbol{n}_{mp}}{2\,\gamma_{mp}}}_{u_{mp}^{*,\text{FV}}} - \theta_{mp}\frac{\Delta F_{mp}}{\gamma_{mp}}$$

- $$\widetilde{u_{mp}}^{\pm} = u_{mp}^{*,\text{FV}} \pm \theta_{mp}\frac{\Delta F_{mp}}{\gamma_{mp}} \quad\Longrightarrow\quad u_{mp}^{*,\text{FV}} = \frac{1}{2}\left(\widetilde{u_{mp}}^{-} + \widetilde{u_{mp}}^{+}\right)$$



(a) 1st-order situation (b) Blended flux situation

Riemann problem and entropic weak solution



- $\partial_t\, u(\boldsymbol{x},t) + \nabla_x \,.\, \boldsymbol{F}\,(u(\boldsymbol{x},t)) = 0, \qquad (\boldsymbol{x},t) \in \mathbb{R}^d \times \mathbb{R}^+$
- $u(\boldsymbol{x},0) = \begin{cases} u^- & \text{if } (\boldsymbol{x}\,.\,\boldsymbol{n}) < 0 \\ u^+ & \text{if } (\boldsymbol{x}\,.\,\boldsymbol{n}) > 0 \end{cases}$
- $u(\boldsymbol{x},t) = \mathcal{W}\left(\frac{\boldsymbol{x}\,.\,\boldsymbol{n}}{t};\, u^-,\, u^+\right)$ denotes unique entropic weak solution
- **This solution ensures the following properties:**
  - $\boxed{\mathcal{W}\left(\frac{\boldsymbol{x}\,.\,\boldsymbol{n}}{t};\, u^-,\, u^+\right) \in \mathrm{I}\,(u^-, u^+)}$ maximum principle or positivity
  - For any entropy - entropy flux pair (η, ϕ) entropic inequality

$$\boxed{\frac{1}{2\,\gamma\,\Delta t}\int_{-\gamma\,t}^{\gamma\,t} \eta\Big(\mathcal{W}\Big(\frac{\boldsymbol{x}\,.\,\boldsymbol{n}}{\Delta t};\, u_L,\, u_R\Big)\Big)\,\mathrm{d}\,\xi_n \leq \frac{\eta(u_L)+\eta(u_R)}{2} - \frac{(\phi(u_R)-\phi(u_L))\,.\,\boldsymbol{n}}{2\,\gamma}}$$

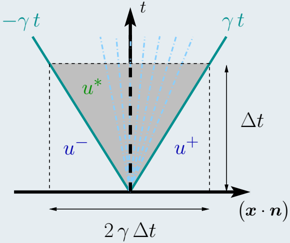
under the condition that $\gamma \geq \max\limits_{w\,\in\,\mathrm{I}\,(u^-,u^+)} \Big(\big|\boldsymbol{F}'(w)\,.\,\boldsymbol{n}\big|\Big)$

Riemann intermediate state and entropic weak solution average

- Taking γ such that $\gamma \geq \max\limits_{w \in \mathrm{I}(u^-,u^+)} \left(\left| \boldsymbol{F}'(w) \,.\, \boldsymbol{n} \right| \right)$

- $$u^* := \frac{u^- + u^+}{2} - \frac{\left(\boldsymbol{F}(u^+) - \boldsymbol{F}(u^-)\right)}{2\,\gamma} \,.\, \boldsymbol{n}$$

$$= \frac{1}{2\,\gamma\,\Delta t} \int_{-\gamma\,\Delta t}^{\gamma\,\Delta t} \mathcal{W}\left(\frac{\boldsymbol{x}\,.\,\boldsymbol{n}}{\Delta t};\, u^-,\, u^+\right) \mathrm{d}x$$



- **This Riemann intermediate state ensures the following properties:**
  - $u^* \in \mathrm{I}(u^-, u^+)$ — maximum principle or positivity
  - For any entropy - entropy flux pair (η, ϕ) — entropic inequality

$$\eta\,(u^*) \leq \frac{\eta(u_L) + \eta(u_R)}{2} - \frac{\left(\phi(u_R) - \phi(u_L)\right)\,.\,\boldsymbol{n}}{2\,\gamma}$$

Admissible solution — Find the correct θ_{mp}

- G a convex admissible set where the solution has to remain in
  - $G = \left\{ u_0 \in [\alpha, \beta] \implies u \in [\alpha, \beta] \right\}$ — SCL
  - $G = \left\{ u = \begin{pmatrix} h \\ \boldsymbol{q} \end{pmatrix}, \quad h \geq 0 \right\}$ — NSW
  - $G = \left\{ u = \begin{pmatrix} \rho \\ \boldsymbol{q} \\ E \end{pmatrix}, \quad \rho > 0,\ p(u) > 0 \right\}$ — Euler
- **A sufficient condition to ensure $\overline{u}_m^{c,n+1} \in G$ is that**

$$\forall\, S_p^v \in \mathcal{V}_m^c, \qquad \widetilde{u_{mp}}^{\pm} = u_{mp}^{*,\,\mathrm{FV}} \pm \theta_{mp} \frac{\Delta F_{mp}}{\gamma_{mp}} \in G$$

- We want to prevent to code from crashing (apparition of NaN, . . .)
- We want to prevent the apparition of spurious oscillations
- ***We want to ensure discrete entropy inequalities (?)***

Questions regarding entropy

- **Can we find the θ_{mp} coefficients ensuring an entropy inequality?**
- **What do we mean by entropy inequality?**
  - for one or any entropy?
  - at the discrete or semi-discrete time level?
  - at the cells or subcells space level?
- **If we manage to ensure an entropy inequality, is it worth the effort?**
  - in terms of accuracy
  - in terms of other critical properties to ensure, as positivity for instance
- **Do we really need an entropy inequality to practically capture the entropic weak solution?**
- **If numerical diffusion is the key, how much do we need?**

Definitions

- (η, ϕ) — entropy - entropy flux
- $v(u) = \eta'(u)$ — entropy variable
- $\psi(u) = v(u)\,\boldsymbol{F}(u) - \phi(u)$ — entropy potential flux

Subcell entropy stability at the discrete level — **for all** (η, ϕ)

- if $\Delta F_{mp} \,.\, \left(\overline{u}_p^{v,n} - \overline{u}_m^{c,n}\right) > 0$,

$$\theta_{mp} \leq \min\left(1,\ \frac{\left(\gamma_{mp} - \gamma_{\max}\right)\left(\overline{u}_p^{v,n} - \overline{u}_m^{c,n}\right)}{2\,\Delta F_{mp}}\right)$$

- $\gamma_{\max} \quad := \max\limits_{w \,\in\, \mathrm{I}\,(\overline{u}_m^{c,n},\overline{u}_p^{v,n})} \left(\left|\boldsymbol{F}'(w) \,.\, \boldsymbol{n}_{mp}\right|\right)$

$\Longrightarrow$ **1st order scheme!**

Semi-discrete subcell entropy dissipation **for a given** (η, ϕ)

- if $\Delta F_{mp} \cdot \left(v\left(\overline{u}_p^{v,n}\right) - v\left(\overline{u}_m^{c,n}\right) \right) > 0$,

$$\theta_{mp} \leq \min \left(1, \frac{\left(\frac{\psi\left(\overline{u}_p^{v,n}\right) - \psi\left(\overline{u}_m^{c,n}\right)}{v\left(\overline{u}_p^{v,n}\right) - v\left(\overline{u}_m^{c,n}\right)} \right) \cdot \boldsymbol{n}_{mp} - \mathcal{F}_{mp}^{\mathrm{FV}}}{\Delta F_{mp}} \right)$$

$\Longrightarrow$ **2nd order scheme!**

A. Rueda-Ramírez, B. Bolm, D. Kuzmin and **G. Gassner**, *Monolithic Convex Limiting for Legendre-Gauss-Lobatto Discontinuous Galerkin Spectral Element Methods.* Arxiv, 2023.

Histopolation and sub-resolution basis functions $N_s = N_k$

- Let $\{\lambda_m^c\}_m$ be the histopolation basis function such that, for $v_h^c \in \mathbb{P}^k(\omega_c)$

$$v_h^c = \sum_{m=1}^{N_k} \overline{v}_m^c \, \lambda_m^c$$

- Let $\{\varphi_m^c\}_m$ be the sub-resolution basis function such that, $\forall \psi \in \mathbb{P}^k(\omega_c)$

$$\int_{\omega_c} \varphi_m \, \psi \, \mathrm{d}V = \int_{S_m^c} \psi \, \mathrm{d}V$$

- Then, given $v_h^c \in \mathbb{P}^k(\omega_c)$, it writes

$$v_h^c = \sum_{m=1}^{N_k} \underline{v}_m^c \, \varphi_m^c$$

Orthogonality property

$$\boxed{\int_{\omega_c} \lambda_m^c \, \varphi_p^c \, \mathrm{d}V = |S_m^c| \, \delta_{mp}}$$

Semi-discrete cell entropy dissipation **for a given** (η, ϕ)

- $$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\omega_c} \eta(u_h^c)\, \mathrm{d}V = \oint_{\omega_c} v(u_h^c)\, \partial_t u_h^c\, \mathrm{d}V = \int_{\omega_c} v_h^c\, \partial_t u_h^c\, \mathrm{d}V \equiv \Delta\eta_c$$

- $v_h^c = \sum_{m=1}^{N_k} \underline{v}_m^c\, \varphi_m^c$ $\quad$ L^2 projection of $v(u_h^c)$ onto $\mathbb{P}^k$

- $$\Delta\eta_c = \int_{\omega_c} \left(\sum_{m=1}^{N_k} \underline{v}_m^c\, \varphi_m^c \right) \left(\sum_{m=1}^{N_k} \frac{\mathrm{d}\,\overline{u}_m^c}{\mathrm{d}t}\, \lambda_m^c \right) \mathrm{d}V = \sum_{m=1}^{N_k} |S_m^c|\, \underline{v}_m^c\, \frac{\mathrm{d}\,\overline{u}_m^c}{\mathrm{d}t}$$

For sake of simplicity, let us consider 1D $\quad N_k = k+1$

- $$\Delta\eta_i = -\sum_{m=1}^{k+1} \underline{v}_m^i \left(\widetilde{F}_{m+\frac{1}{2}} - \widetilde{F}_{m-\frac{1}{2}} \right) = \mathbf{A}_{vol} + \mathbf{A}_{bdr}$$

- $$\mathbf{A}_{vol} = \sum_{m=1}^{k+1} \left(\underline{v}_{m+1}^i - \underline{v}_m^i \right) \widetilde{F}_{m+\frac{1}{2}} + \left(\underline{v}_1^i - v\left(u_h^i(x_{i-\frac{1}{2}})\right) \right) \theta_{\frac{1}{2}}^i\, \widehat{F}_{\frac{1}{2}}^i + \left(v\left(u_h^i(x_{i+\frac{1}{2}})\right) - \underline{v}_{k+1}^i \right) \theta_{k+\frac{3}{2}}^i\, \widehat{F}_{k+\frac{3}{2}}^i$$

Boundary entropy contribution

- $\mathcal{F}^{\text{FV}}_{i-\frac{1}{2}} \equiv \mathcal{F}^{i,\text{FV}}_{\frac{1}{2}}, \quad \mathcal{F}^{\text{FV}}_{i+\frac{1}{2}} \equiv \mathcal{F}^{i,\text{FV}}_{k+\frac{3}{2}}, \quad \mathcal{F}^{\text{DG}}_{i-\frac{1}{2}} \equiv \widehat{F}^{i}_{\frac{1}{2}}, \quad \mathcal{F}^{\text{DG}}_{i+\frac{1}{2}} \equiv \widehat{F^{i}_{k+\frac{3}{2}}}$
- $\theta_{i+\frac{1}{2}} \equiv \theta^{i}_{k+\frac{3}{2}} = \theta^{i+1}_{\frac{1}{2}}$
- $v^{\mp}_{i\pm\frac{1}{2}} \equiv v\big(u^{i}_{h}(x_{i\pm\frac{1}{2}})\big)$
- $$\mathbf{A}_{bdr} = \underline{v}^{i}_{1}\left(1-\theta_{i-\frac{1}{2}}\right)\mathcal{F}^{\text{FV}}_{i-\frac{1}{2}} + v^{+}_{i-\frac{1}{2}}\,\theta_{i-\frac{1}{2}}\,\mathcal{F}^{\text{DG}}_{i-\frac{1}{2}}$$
$$-\underline{v}^{i}_{k+1}\left(1-\theta_{i+\frac{1}{2}}\right)\mathcal{F}^{\text{FV}}_{i+\frac{1}{2}} - v^{-}_{i+\frac{1}{2}}\,\theta_{i+\frac{1}{2}}\,\mathcal{F}^{\text{DG}}_{i+\frac{1}{2}}$$

Semi-discrete cell entropy stability **for a given** (η,ϕ)

- $\mathcal{F}^{\text{FV}}_{m+\frac{1}{2}} := \mathcal{F}\big(u(\underline{v}_{m+1}), u(\underline{v}_{m})\big)$ modified FV numerical flux
- A sufficient condition to entropy stability writes as follows

$$\mathbf{A}_{vol} \leq \theta_{i-\frac{1}{2}}\Big(\psi(\underline{v}^{i}_{1}) - \psi(v^{+}_{i-\frac{1}{2}})\Big) + \theta_{i+\frac{1}{2}}\Big(\psi(v^{-}_{i+\frac{1}{2}}) - \psi(\underline{v}^{i}_{k+1})\Big) + \psi(\underline{v}^{i}_{k+1}) - \psi(\underline{v}^{i}_{1})$$

Y. Lin and J. Chan, *High order entropy stable discontinuous Galerkin spectral element methods through subcell limiting.* JCP, 2024.

Knapsack problem

- The sufficient condition rewrites as

$$\boxed{\boldsymbol{a} \,.\, \boldsymbol{\Theta} \leq b}$$

- $\boldsymbol{\Theta} = \left(\theta^i_{\frac{1}{2}}, \, \dots, \, \theta^i_{k+\frac{3}{2}}\right)^{\mathrm{t}}$
- $\boldsymbol{a} = \left(a^i_{\frac{1}{2}}, \, \dots, \, a^i_{k+\frac{3}{2}}\right)^{\mathrm{t}}$ defined as

$$\begin{cases} a_{\frac{1}{2}} = \left(\underline{v}^i_1 - v^+_{i-\frac{1}{2}}\right) \mathcal{F}^{\mathrm{DG}}_{i-\frac{1}{2}} - \Big(\psi\big(\underline{v}^i_1\big) - \psi\big(v^+_{i-\frac{1}{2}}\big)\Big), \\ a_{m+\frac{1}{2}} = \left(\underline{v}^i_{m+1} - \underline{v}^i_m\right) \left(\widehat{F_{m+\frac{1}{2}}} - \mathcal{F}^{\mathrm{FV}}_{m+\frac{1}{2}}\right), \qquad m \in [\![1, \, k]\!] \\ a_{k+\frac{3}{2}} = \left(v^-_{i+\frac{1}{2}} - \underline{v}^i_{k+1}\right) \mathcal{F}^{\mathrm{DG}}_{i+\frac{1}{2}} - \Big(\psi\big(v^-_{i+\frac{1}{2}}\big) - \psi\big(\underline{v}^i_{k+1}\big)\Big), \end{cases}$$

- $b = \psi\big(\underline{v}^i_{k+1}\big) - \psi\big(\underline{v}^i_1\big) - \sum_{m=1}^{k+1} \left(\underline{v}^i_{m+1} - \underline{v}^i_m\right) \mathcal{F}^{\mathrm{FV}}_{m+\frac{1}{2}}$
- **Because $b \geq 0$ the Knapsack problem is indeed solvable**

Greedy algorithm

- Find $\mathbf{0} \leq \Theta \leq \Theta_C \leq \mathbf{1}$ maximizing $\sum_{m=1}^{k+1} \theta_{m+\frac{1}{2}}$ such that $\boxed{\boldsymbol{a} \,.\, \Theta \leq b}$
- $\Theta_C = \left(\theta^C_{\frac{1}{2}}, \ldots, \theta^C_{k+\frac{3}{2}}\right)^{\mathrm{t}}$ is a given supplementary constraint

High-order accuracy preservation

- Let us consider u a smooth exact solution
- $u_h^i = u + O(\Delta x^{k+1})$ $\quad$ $(k+1)^{\text{th}}$-order approximation
- Then, we have that

$$\boxed{\boldsymbol{a} \,.\, \mathbf{1} - b = O(\Delta x^{k+2})}$$

- This implies that

$$\boxed{\widetilde{F^i_{m+\frac{1}{2}}} = F\big(u(x^i_{m+\frac{1}{2}})\big) + O(\Delta x^{k+1})}$$

Linear advection of a composite signal $\eta(u) = \frac{1}{2}u^2$

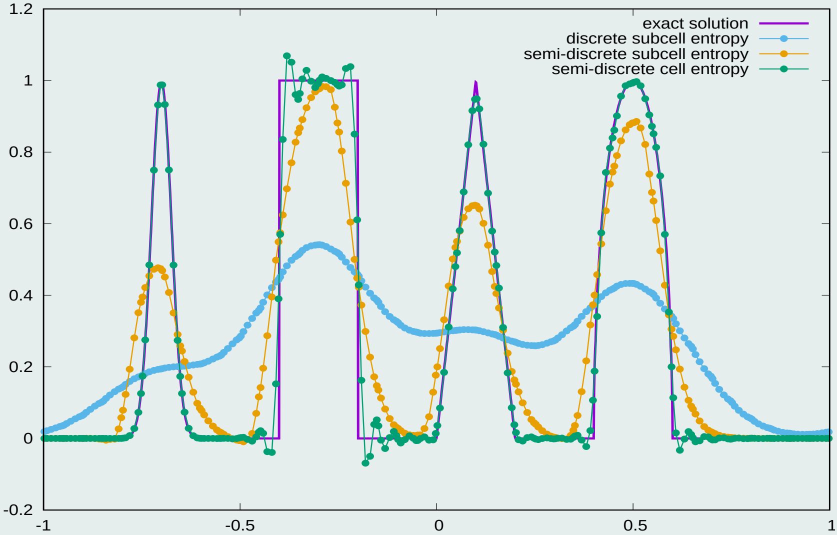


Figure: $\mathbb{P}^5$-DG/FV solutions on 40 cells: submean values

Linear advection of a composite signal $\eta(u) = |u - k_e|^{1+\epsilon} \backslash (1+\epsilon)$

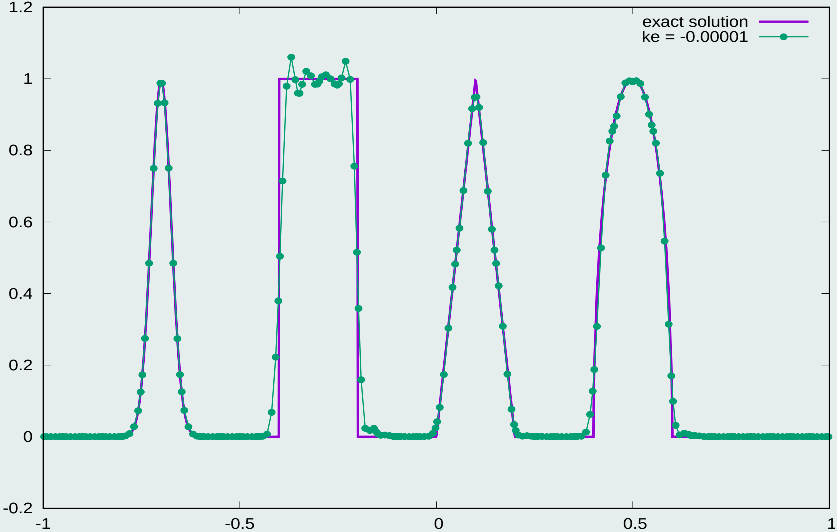


Figure: $\mathbb{P}^5$-DG/FV submean values on 40 cells: $\epsilon = 0.25$ and k_e =-1.D-5

Linear advection of a composite signal $\eta(u) = |u - k_e|^{1+\epsilon} \backslash (1+\epsilon)$

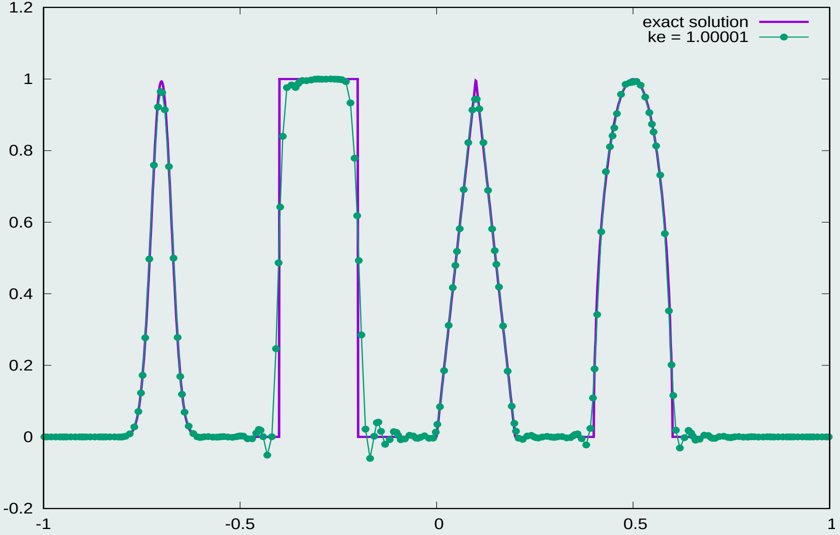


Figure: $\mathbb{P}^5$-DG/FV submean values on 40 cells: $\epsilon = 0.25$ and $k_e = 1 + 1.D\text{-}5$

Non-linear non-convex flux Buckley case 80 cells

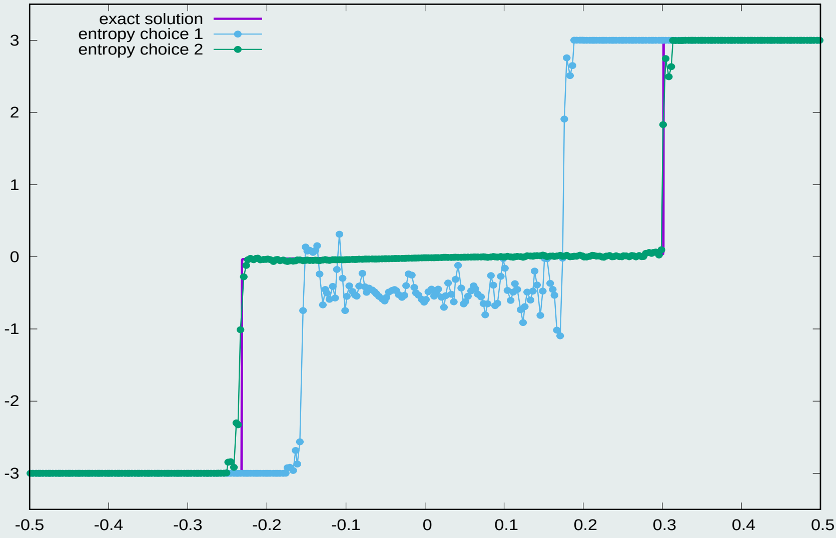


Figure: $\mathbb{P}^3$-DG/FV submean values: $\eta_1(u) = \frac{1}{2}u^2$ and $\eta_2(u) = \int \text{atan}(20u)\,\mathrm{d}u$

Non-linear non-convex flux Buckley case 80 cells

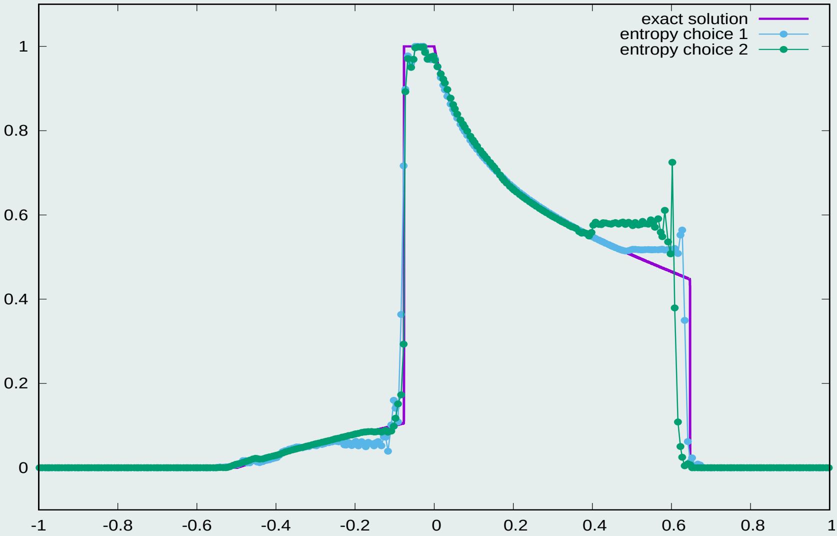
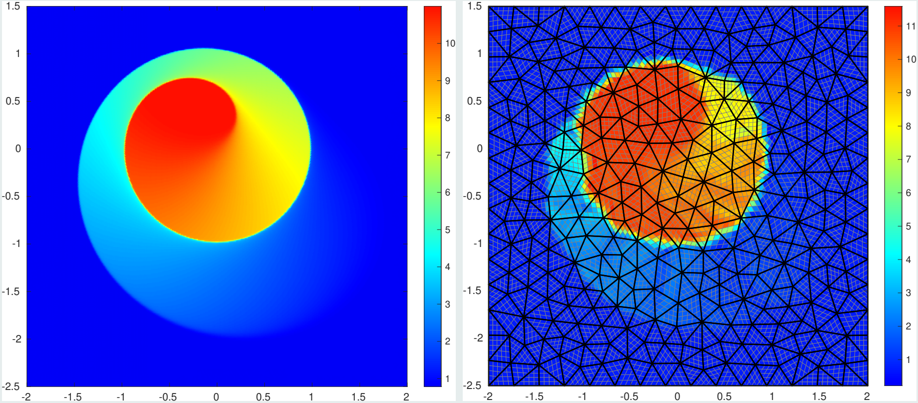


Figure: $\mathbb{P}^3$-DG/FV submean values: $\eta_1(u) = \frac{1}{2}u^2$ and $\eta_2(u) = \int \mathrm{atan}(20u)\,\mathrm{d}u$

KPP non-convex flux problem $\eta(u) = \frac{1}{2}u^2$



(a) 1^{th}-order FV on 209184 cells (b) $\mathbb{P}^4$-DG/FV on 576 cells

Figure: $\mathbb{P}^4$-DG/FV entropic scheme: non-entropic solution

Questions regarding entropy $\longrightarrow$ pieces of answer

- **Can we find $\theta^i_{m+\frac{1}{2}}$ coefficients ensuring an entropy inequality?**

  $\hookrightarrow$ **Yes!**

- **What do we mean by entropy inequality, and is it worth the effort?**
  - for any entropy, at the discrete time level and for any subcell

    $\hookrightarrow$ **1^{st}-order**
  - for a given entropy, at the semi-discrete time level for any subcells

    $\hookrightarrow$ **2^{nd}-order**
  - for a given entropy, at the semi-discrete time level for any cells

    $\hookrightarrow$ **$(k+1)^{th}$-order**

    $\hookrightarrow$ $\mathcal{F}^{FV}_{m+\frac{1}{2}} := \mathcal{F}(u(\underline{v}_{m+1}), u(\underline{v}_m))$

- **Do we need entropy stability or just *"enough"* numerical diffusion?**

  $\hookrightarrow$ **Unclear...** $\Longrightarrow$ **GMP and LMP + relaxation**

Global maximum principle $\overline{u}_m^{c,n+1} \in [\alpha,\ \beta]$

$$\theta_{mp} \leq \min\left(1,\ \left|\frac{\gamma_{mp}}{\Delta F_{mp}}\right| \min\left(\beta - u_{mp}^{*,\,\mathrm{FV}},\ u_{mp}^{*,\,\mathrm{FV}} - \alpha\right)\right)$$

Local maximum principle $\overline{u}_m^{c,n+1} \in [\alpha_m^c,\ \beta_m^c]$

- $\alpha_m^c := \min\limits_{S_q^w \in \mathcal{N}(S_m^c)} \left(\overline{u}_q^{w,n}\right)$ and $\beta_m^c := \max\limits_{S_q^w \in \mathcal{N}(S_m^c)} \left(\overline{u}_q^{w,n}\right)$

$$\theta_{mp} \leq \min\left(1,\ \left|\frac{\gamma_{mp}}{\Delta F_{mp}}\right| \begin{cases} \min\left(\beta_p^v - u_{mp}^{*,\,\mathrm{FV}},\ u_{mp}^{*,\,\mathrm{FV}} - \alpha_m^c\right) & \text{if } \Delta F_{mp} > 0 \\ \min\left(\beta_m^c - u_{mp}^{*,\,\mathrm{FV}},\ u_{mp}^{*,\,\mathrm{FV}} - \alpha_p^v\right) & \text{if } \Delta F_{mp} < 0 \end{cases}\right)$$

- The wider set $\mathcal{N}(S_m^c)$ is, the softer this local maximum principle will be
- **Smooth extrema relaxation to preserve accuracy**

Linear advection of a composite signal — 40 cells

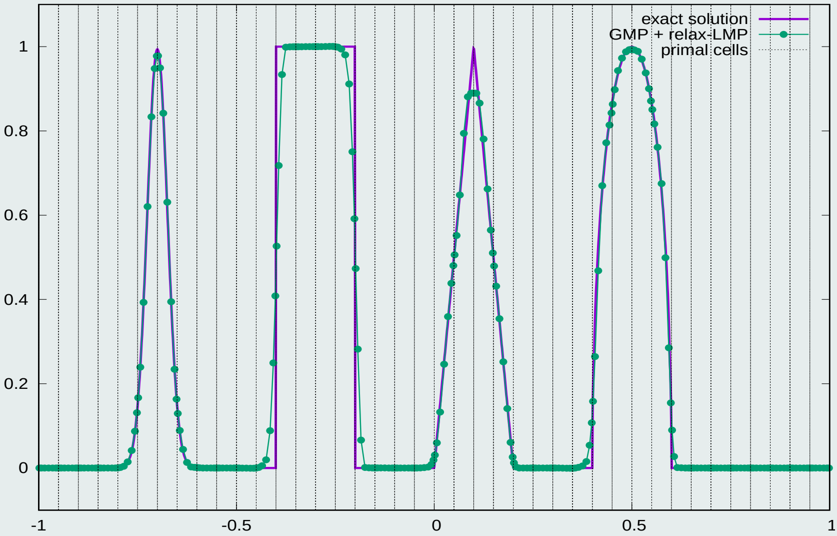


Figure: $\mathbb{P}^6$-DG/FV with GMP and relaxed-LMP: submean values

Linear advection of a composite signal 40 cells

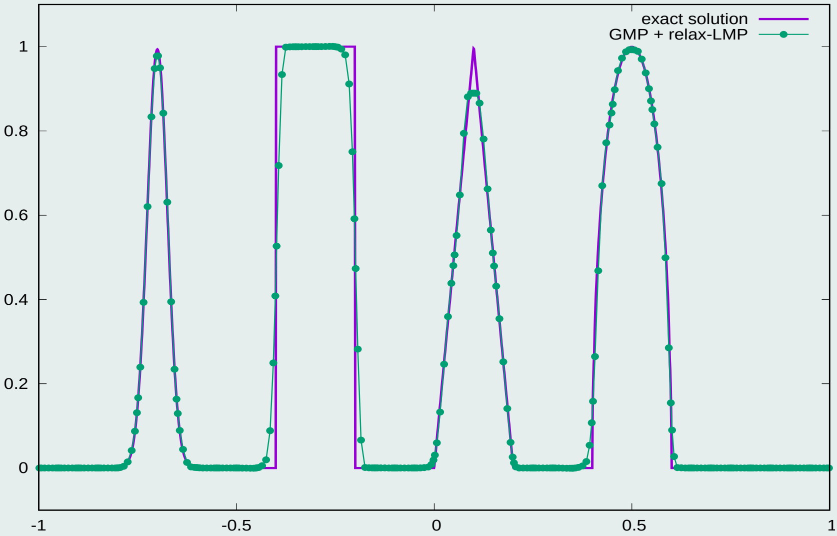


Figure: $\mathbb{P}^6$-DG/FV with GMP and relaxed-LMP: submean values

Non-linear non-convex flux Buckley case 40 cells

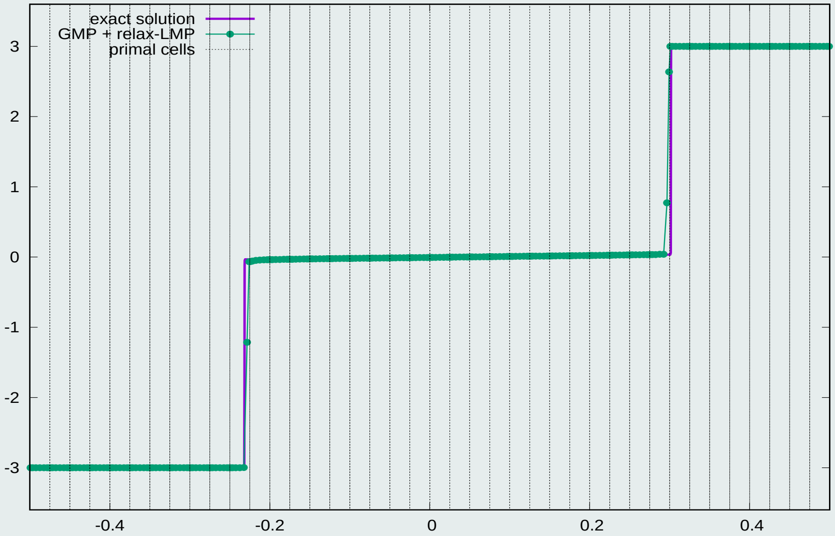


Figure: $\mathbb{P}^6$-DG/FV with GMP and relaxed-LMP: submean values

Non-linear non-convex flux Buckley case 40 cells

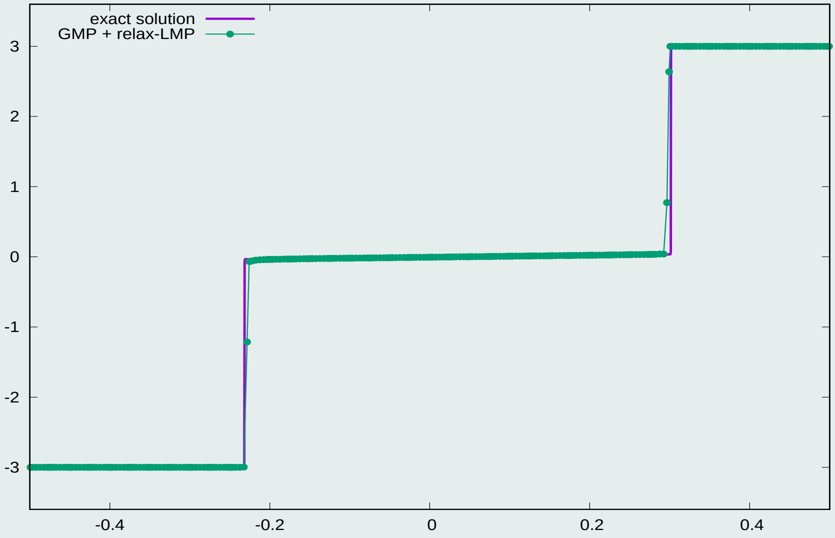


Figure: $\mathbb{P}^6$-DG/FV with GMP and relaxed-LMP: submean values

Non-linear non-convex flux Buckley case 40 cells

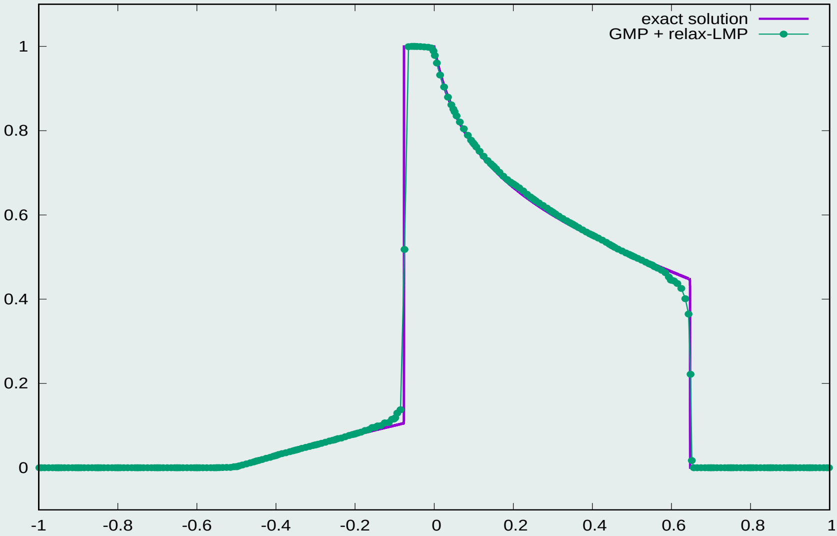
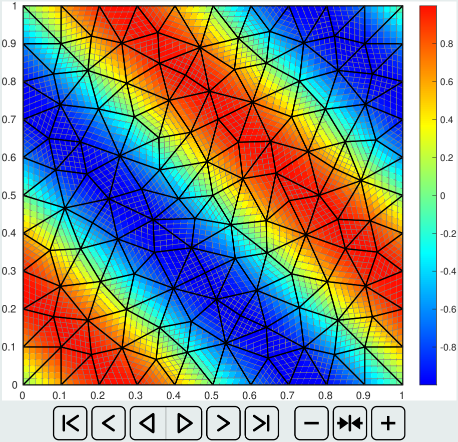
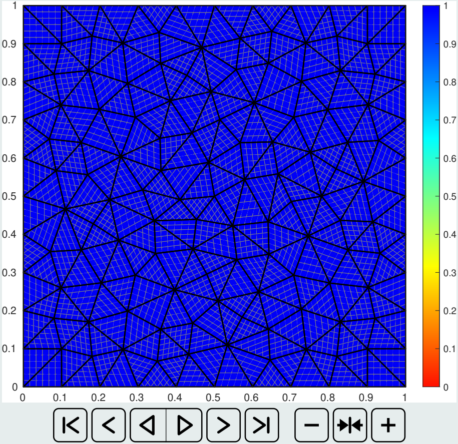


Figure: $\mathbb{P}^6$-DG/FV with GMP and relaxed-LMP: submean values

Burgers equation $u_0(x, y) = \sin(2\pi\,(x + y))$



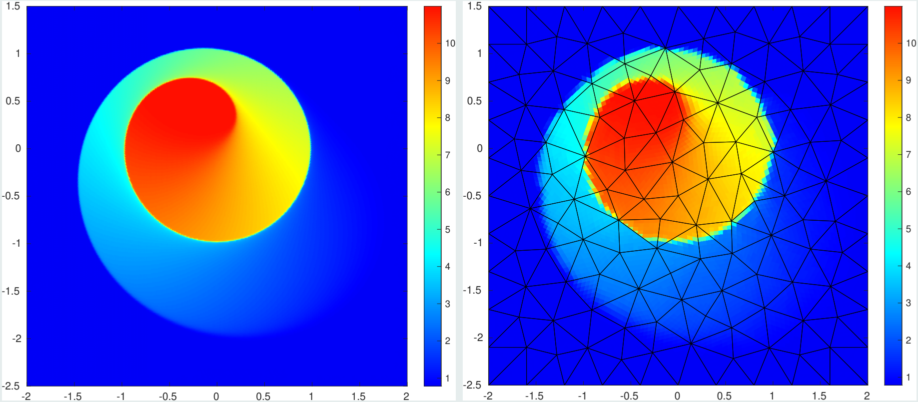
(a) Solution submean values



(b) Blending coefficients

Figure: $\mathbb{P}^5$-DG/FV scheme with GMP and relaxed-LMP on 242 cells

KPP non-convex flux problem



(a) 1^{th}-order FV on 209184 cells

(b) $\mathbb{P}^7$-DG/FV on 272 cells

Figure: $\mathbb{P}^7$-DG/FV scheme with GMP and relaxed-LMP

Non-linear Euler compressible gas dynamics equations

- $\partial_t \boldsymbol{V} + \nabla_x \,.\, \boldsymbol{F}(\boldsymbol{V}) = \boldsymbol{0}$
- $\boldsymbol{V} = \begin{pmatrix} \rho \\ \boldsymbol{q} \\ E \end{pmatrix}$ <u>conservative variables</u>
- $\boldsymbol{F}(\boldsymbol{V}) = \begin{pmatrix} \boldsymbol{q} \\ \dfrac{\boldsymbol{q} \otimes \boldsymbol{q}}{\rho} + p\, I_d \\ (E + p)\, \dfrac{\boldsymbol{q}}{\rho} \end{pmatrix}$ <u>flux function</u>
- $p := p(\boldsymbol{V}) = (\gamma - 1) \left(E - \dfrac{1}{2} \dfrac{\|\boldsymbol{q}\|^2}{\rho} \right)$ <u>equation of state</u>

Monolithic subcell DG/FV scheme property

- Positivity of the density and internal energy, at the subcell scale

Definitions

- $\widetilde{\boldsymbol{F}_{mp}} := \mathcal{F}_{mp}^{\,\mathrm{FV}} + \Theta_{mp}\underbrace{\left(\widehat{\boldsymbol{F}_{mp}} - \mathcal{F}_{mp}^{\,\mathrm{FV}}\right)}_{\Delta \boldsymbol{F}_{mp}}$ convex blended flux

- $\Theta_{mp} = \begin{pmatrix} \theta_{mp}^{\rho} & \mathbf{0} & 0 \\ \mathbf{0} & \boldsymbol{\theta}_{mp}^{\boldsymbol{q}} & \mathbf{0} \\ 0 & \mathbf{0} & \theta_{mp}^{E} \end{pmatrix}$

- $\mathcal{F}_{mp}^{\,\mathrm{FV}} := \mathcal{F}\left(\overline{\boldsymbol{V}}_m^{c,n}, \overline{\boldsymbol{V}}_p^{v,n}, \boldsymbol{n}_{mp}\right)$ Global L-F, Rusanov, HLL(C), ...

Positivity of the density

- $\theta_{mp}^{\rho} = \theta_{mp}^{(1)}\,\theta_{mp}^{(2)}$

$$\theta_{mp}^{(1)} \leq \min\left(1, \left|\frac{\gamma_{mp}}{\Delta F_{mp}^{\rho}}\right| \rho_{mp}^{*,\,\mathrm{FV}}\right)$$

Positivity of the internal energy

- $A_{mp} = \dfrac{1}{(\gamma_{mp})^2} \left(\dfrac{1}{2} \left\| \Delta \boldsymbol{F}^{\boldsymbol{q}}_{mp} \right\|^2 - \theta^{(1)}_{mp} \, \Delta F^{\rho}_{mp} \, \Delta F^{E}_{mp} \right)$
- $B_{mp} = \dfrac{1}{\gamma_{mp}} \left(\boldsymbol{q}^{*,\,\mathrm{FV}}_{mp} \cdot \Delta \boldsymbol{F}^{\boldsymbol{q}}_{mp} - \rho^{*,\,\mathrm{FV}}_{mp} \, \Delta F^{E}_{mp} - \theta^{(1)}_{mp} \, E^{*,\,\mathrm{FV}}_{mp} \, \Delta F^{\rho}_{mp} \right)$
- $M_{mp} = \rho^{*,\,\mathrm{FV}}_{mp} \, E^{*,\,\mathrm{FV}}_{mp} - \frac{1}{2} \left\| \boldsymbol{q}^{*,\,\mathrm{FV}}_{mp} \right\|^2$

$$\boxed{\theta^{(2)}_{mp} \leq \min \left(1, \, \frac{M_{mp}}{\left| B_{mp} \right| + \max\left(0, A_{mp}\right)} \right)}$$

- $\theta^{\rho}_{mp} = \theta^{\rho\,(1)}_{mp} \, \theta^{(2)}_{mp}, \quad \theta^{q^x}_{mp} = \theta^{(2)}_{mp}, \quad \theta^{q^y}_{mp} = \theta^{(2)}_{mp}, \quad \theta^{E}_{mp} = \theta^{(2)}_{mp}$

A. Rueda-Ramírez, **B. Bolm**, **D. Kuzmin** and **G. Gassner**, *Monolithic Convex Limiting for Legendre-Gauss-Lobatto Discontinuous Galerkin Spectral Element Methods.* Arxiv, 2023.

LMP $\overline{v}_m^{c,n+1} \in [\alpha_m^c, \beta_m^c]$

- $v \in \{\rho, q^x, q^y, E\}$ conservative variable
- $\alpha_m^c := \min\limits_{S_q^w \in \mathcal{N}(S_m^c)} \left(\overline{v}_q^{w,n}, v_{mq}^{*,\text{FV}}\right)$ and $\beta_m^c := \max\limits_{S_q^w \in \mathcal{N}(S_m^c)} \left(\overline{u}_q^{w,n}, v_{mq}^{*,\text{FV}}\right)$

$$\theta_{mp} \leq \min\left(1, \left|\frac{\gamma_{mp}}{\Delta F_{mp}}\right| \begin{cases} \min\left(\beta_p^v - u_{mp}^{*,\text{FV}}, u_{mp}^{*,\text{FV}} - \alpha_m^c\right) & \text{if } \Delta F_{mp} > 0 \\ \min\left(\beta_m^c - u_{mp}^{*,\text{FV}}, u_{mp}^{*,\text{FV}} - \alpha_p^v\right) & \text{if } \Delta F_{mp} < 0 \end{cases}\right)$$

- **Smooth extrema relaxation to preserve accuracy**

Sod shock tube test case — 10 cells

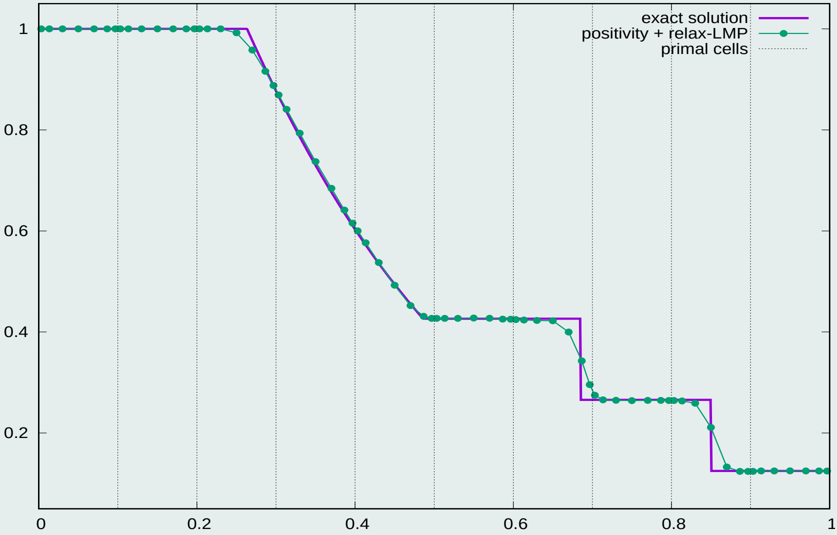


Figure: $\mathbb{P}^6$-DG/FV scheme with GMP and relaxed-LMP: submean values

Double rarefaction test case — 10 cells

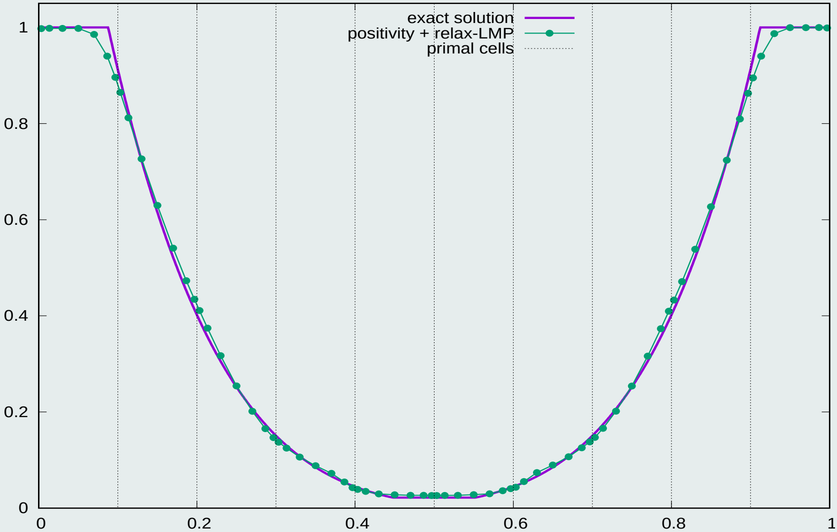


Figure: $\mathbb{P}^6$-DG/FV scheme with GMP and relaxed-LMP: submean values

Shock acoustic-wave interaction test case 200 cells

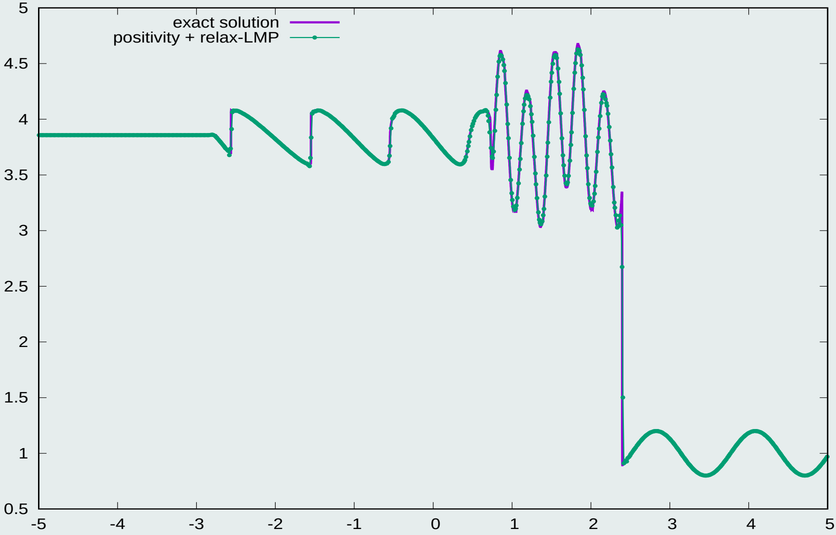
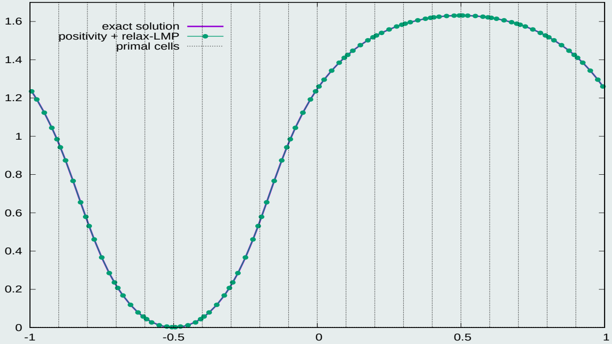


Figure: $\mathbb{P}^3$-DG/FV scheme with GMP and relaxed-LMP: HLL-C numerical flux

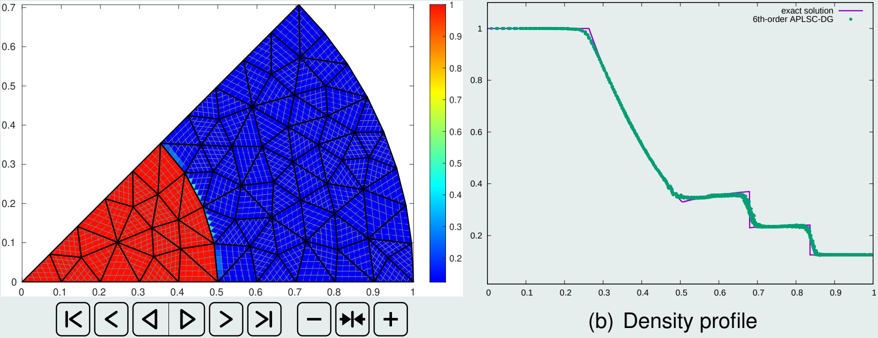
Smooth isentropic solution $\rho_0 = 1 + 0.9999999\sin(2\pi x)$



	L_1		L_2	
h	$E^h_{L_1}$	$q^h_{L_1}$	$E^h_{L_2}$	$q^h_{L_2}$
$\frac{1}{20}$	1.54E-5	4.01	2.04E-5	3.82
$\frac{1}{40}$	9.57E-7	4.89	1.45E-6	4.85
$\frac{1}{80}$	3.22E-8	4.84	5.00E-8	4.87
$\frac{1}{160}$	1.12E-9	-	1.71E-9	-

Table: Convergence rates computed on the pressure with a 5th-order DG/FV scheme

Sod shock tube problem in cylindrical geometry

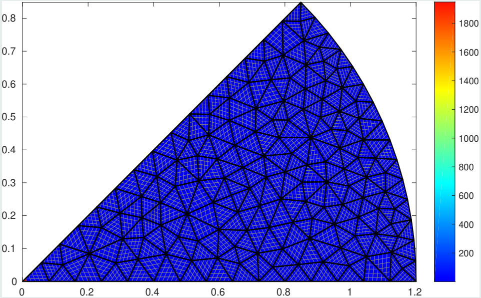


(a) Density map

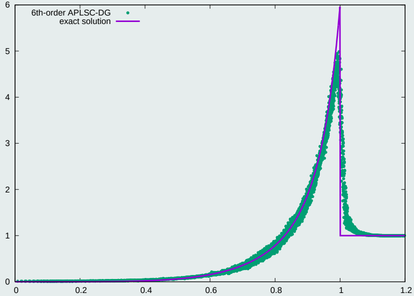
(b) Density profile

Figure: 6th-order DG/FV with GMP and relaxed-LMP on a 110 cells mesh

Sedov point blast problem in cylindrical geometry



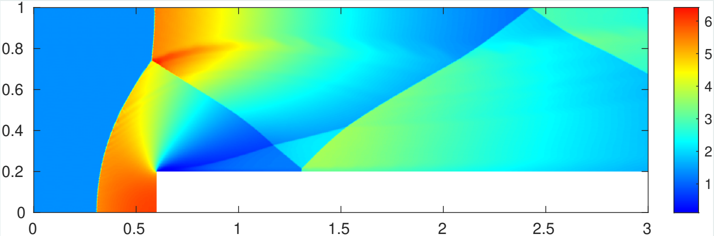
(a) Energy map



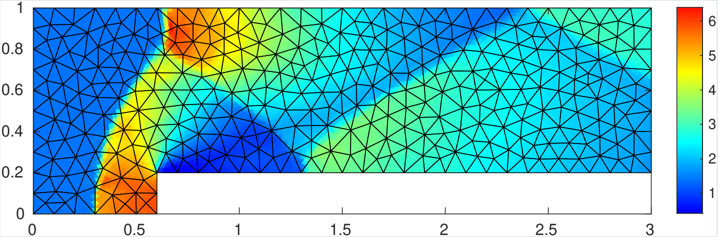
(b) Density profile

Figure: 6th-order DG/FV on a 271 cells mesh at $t = 1$

Mach 3 forward-facing step



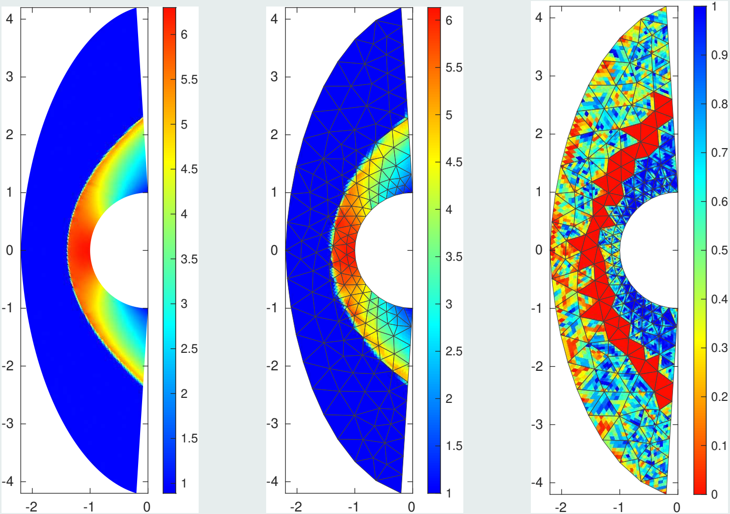
(a) $\mathbb{P}^1$ on 84108 cells



(b) $\mathbb{P}^3$ on 680 cells

Figure: Monolithic subcell DG/FV scheme: subcell density mean values

Mach 20 hypersonic flow over half cylinder



(a) $\mathbb{P}^0$ on 25266 cells (b) $\mathbb{P}^5$ on 292 cells (c) $\mathbb{P}^5$ on 292 cells

Figure: Monolithic subcell DG/FV scheme: density and blending coefficients

Monolithic local subcell DG/FV scheme

- Reformulate DG schemes as subgrid FV-like schemes:
  - regardless the type of mesh used
  - regardless the space dimension (***in theory***...)
  - regardless the cell subdivision ($N_s \geq N_k$)
- Combine high-order reconstructed fluxes and 1^{st}-order FV fluxes
  - ensuring a maximum or positivity preserving principle at the subcell scale
  - ensuring different entropy stability inequalities
  - reducing significantly the apparition of spurious oscillations
  - preserving the very accurate subcell resolution of DG schemes

Questions

- Is an entropy inequality for one entropy enough?

  $\hookrightarrow$ **Generally, no**
- Is entropy inequality absolutely needed?

  $\hookrightarrow$ **Maybe not** $\Longrightarrow$ **GMP + relaxed LMP**

Articles on this topic

- **F.V**, *A Posteriori Correction of High-Order DG Scheme through Subcell Finite Volume Formulation and Flux Reconstruction.* JCP, 2018.

- **A. Haidar**, **F. Marche** and **F.V**, *A posteriori Finite-Volume local subcell correction of high-order discontinuous Galerkin schemes for the nonlinear shallow-water equations*. JCP, 2022.

- **F.V** and **R. Abgrall**, *A posteriori local subcell correction of DG schemes through Finite Volume reformulation on unstructured grids.* SIAM SISC, 2023.

- **A. Haidar**, **F. Marche** and **F.V**, *Free-boundary problems for wave-structure interactions in shallow-water: DG-ALE description and local subcell correction*. JSC, 2023.

- **A. Haidar**, **F. Marche** and **F.V**, *A robust DG-ALE formulation for nonlinear shallow-water interactions with a floating object*. JSC, 2024.

- **F.V**, *Monolithic local subcell DG/FV convex property preserving scheme on unstructured grids and entropy consideration.* **Under preparation.**

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- **F.V**, *A Posteriori Correction of High-Order DG Scheme through Subcell Finite Volume Formulation and Flux Reconstruction.* JCP, 2018.

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- **A. HAIDAR**, **F. MARCHE** AND **F.V**, *Free-boundary problems for wave-structure interactions in shallow-water: DG-ALE description and local subcell correction*. JSC, 2023.

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One-dimensional case: $N_f^i = N_s - 1$

- $\widehat{\mathrm{F}}_i = -A_i^{\mathrm{t}}\,\mathcal{L}_i^{-1}\left(D_i\,P_i\,M_i^{-1}\,\Phi_i + \mathrm{B}_i\right)$ with $B_i = \left(-\mathcal{F}_{i-\frac{1}{2}}, 0, \dots, 0, \mathcal{F}_{i+\frac{1}{2}}\right)^{\mathrm{t}}$

- $A_i = \begin{pmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & -1 \end{pmatrix}, \quad L_i = A_i\,A_i^{\mathrm{t}} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & -1 & 2 & 1 \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix}$

- $A_i^{\mathrm{t}}\,\mathcal{L}_i^{-1} = \dfrac{1}{N_s}\begin{pmatrix} N_s-1 & -1 & -1 & \dots & \dots & -1 \\ N_s-2 & N_s-2 & -2 & -2 & \dots & -2 \\ N_s-3 & \dots & N_s-3 & -3 & \dots & -3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & -N_s+1 \end{pmatrix}$