Monolithic local subcell DG/FV convex property preserving scheme: entropy consideration

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Scalar conservation law

- $\partial_t u(\mathbf{x},t) + \nabla_x \cdot \mathbf{F}(u(\mathbf{x},t)) = 0, \qquad (\mathbf{x},t) \in \omega \times [0,T]$
- $u(\mathbf{x},0) = u_0(\mathbf{x}), \qquad \mathbf{x} \in \omega$

$(k+1)^{\text{th}}$ order semi-discretization

- $\{\omega_c\}_c$ a partition of ω, such that $\omega = \bigcup_c \omega_c$
- $u_h(\mathbf{x},t)$ the numerical solution, such that $u_{h|\omega_c} = u_h^c \in \mathbb{P}^k(\omega_c)$

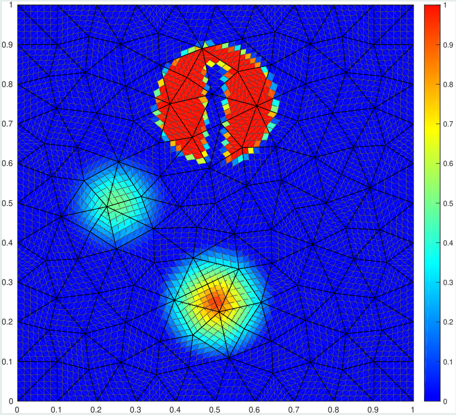
$$u_h^c(\mathbf{x},t) = \sum_{m=1}^{N_k} u_m^c(t)\, \sigma_m^c(\mathbf{x})$$

- $\{\sigma_m^c\}_{m=1,\dots,N_k}$ a basis of $\mathbb{P}^k(\omega_c)$, with $N_k = \frac{(k+1)(k+2)}{2}$ in 2D.

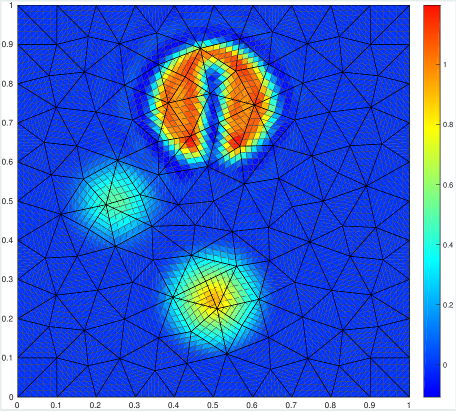
Local variational formulation on ω_c

- $$\int_{\omega_c} \frac{\partial u_h^c}{\partial t}\, \psi \, \mathrm{d}V = \int_{\omega_c} \mathbf{F}(u_h^c) \cdot \nabla_x \psi \, \mathrm{d}V - \int_{\partial\omega_c} \psi\, \mathcal{F}_n \, \mathrm{d}S, \qquad \forall\, \psi \in \mathbb{P}^k(\omega_c)$$
- $\mathcal{F}_n = \mathcal{F}\left(u_h^c, u_h^v, \mathbf{n}\right)$ **numerical flux**

Solid body rotation: discontinuous Galerkin scheme



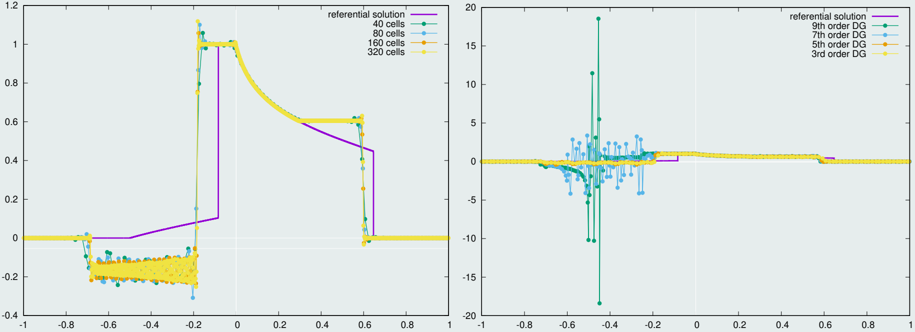
(a) Solution at $t = 0$



(b) Solution at $t = 2\pi$

Figure : Rotation of composite signal on 242 cells: 6[th]-order DG

Spurious oscillations, aliasing and non-entropic behavior



(a) Non-entropic behavior

(b) Aliasing phenomenon

Figure : DG solutions for the Buckley non-convex flux case

Admissible numerical solution

- Maximum principle / positivity preserving
- Ensure a correct entropic behavior

Spurious oscillations

- Discrete maximum principle
- Relaxing condition for smooth extrema

Methodology

Blend, at the subcell scale, high-order DG and 1st-order FV

F. VILAR, *A Posteriori Correction of High-Order DG Scheme through Subcell Finite Volume Formulation and Flux Reconstruction.* JCP, 2018.

F. VILAR AND **R. ABGRALL**, *A posteriori local subcell correction of DG schemes through Finite Volume reformulation on unstructured grids.* SIAM Sci. Comp., 2023.

DG as a subcell Finite Volume

- Rewrite DG scheme as a FV-like scheme on a subgrid

Cell subdivision into $N_S \geq N_k$ subcells

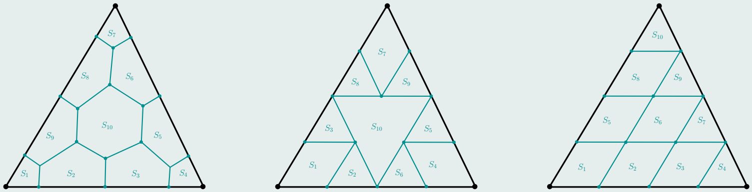


Figure : Examples of $N_s = N_k$ subdivision for $\mathbb{P}^3$ DG scheme on a triangle

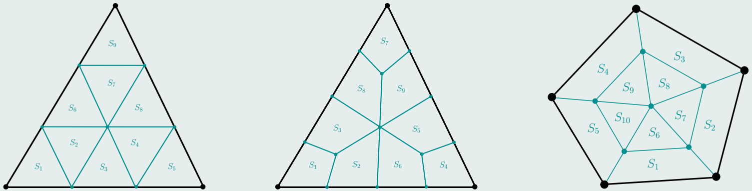


Figure : Examples of $N_S \geq N_k$ subdivision

DG schemes through residuals

- $$\sum_{m=1}^{N_k} \frac{\mathrm{d}\, u_m^c}{\mathrm{d}t} \int_{\omega_c} \sigma_m \sigma_p \,\mathrm{d}V = \int_{\omega_c} \mathbf{F}(u_h^c) \,.\, \nabla_x \sigma_p \,\mathrm{d}V - \int_{\partial\omega_c} \sigma_p \, \mathcal{F}_n \,\mathrm{d}S, \quad \forall p \in [\![1, N_k]\!]$$

$$\Longrightarrow \quad \boxed{M_c \frac{\mathrm{d}\, U_c}{\mathrm{d}t} = \Phi_c}$$

- $(U_c)_m = u_m^c$ — Solution moments
- $(M_c)_{mp} = \int_{\omega_c} \sigma_m \sigma_p \,\mathrm{d}V$ — Mass matrix
- $(\Phi_c)_m = \int_{\omega_c} \mathbf{F}(u_h^c) \,.\, \nabla_x \sigma_m \,\mathrm{d}V - \int_{\partial\omega_c} \sigma_m \, \mathcal{F}_n \,\mathrm{d}S$ — DG residuals

Subdivision and definition

- ω_c is subdivided into N_s subcells S_m^c
- Let us define $\overline{\psi}_m^c = \frac{1}{|S_m^c|} \int_{S_m^c} \psi \,\mathrm{d}V$ the subcell mean value

Submean values

- $\overline{u}_m^c = \frac{1}{|S_m^c|} \sum_{q=1}^{N_k} u_q^c \int_{S_m^c} \sigma_q \, \mathrm{d}V \quad \Longrightarrow \quad \boxed{\overline{U}_c = P_c \, U_c}$

- $(\overline{U}_c)_m = \overline{u}_m^c$ Submean values
- $(P_c)_{mp} = \frac{1}{|S_m^c|} \int_{S_m^c} \sigma_p \, \mathrm{d}V$ Projection matrix

$$\Longrightarrow \quad \boxed{\frac{\mathrm{d}\,\overline{U}_c}{\mathrm{d}t} = P_c \, M_c^{-1} \, \Phi_c}$$

Admissibility of the cell sub-partition into subcells

- $P_c^{\mathrm{t}} \, P_c$ **has to be non-singular**

$$\Longrightarrow \quad \boxed{U_c = (P_c^{\mathrm{t}} \, P_c)^{-1} P_c^{\mathrm{t}} \, \overline{U}_c}$$ Least square procedure

- If $N_s = N_k$, $\overline{U}_c = P_c \, U_c \iff U_c = P_c^{-1} \overline{U}_c$

Subcell Finite Volume: reconstructed fluxes

- Let us introduce the **reconstructed fluxes** such that

$$\frac{\mathrm{d}\,\overline{u}_m^c}{\mathrm{d}t} = -\frac{1}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} \widehat{F_{pm}}$$

- $\mathcal{V}_m^c$ is the set of face neighboring subcells of S_m^c
- We impose that on the boundary of cell ω_c, so for $S_p^v \notin \omega_c$

$$\widehat{F_{pm}} = \int_{f_{mp}^c} \mathcal{F}_n \,\mathrm{d}S \equiv \int_{f_{mp}^c} \mathcal{F}\left(u_h^c,\, u_h^v,\, \boldsymbol{n}_{mp}^c\right)\,\mathrm{d}S$$

- Let $\widehat{F_c}$ be the vector containing all the interior faces reconstructed fluxes
- Then, $\widehat{F_c}$ is uniquely defined as following

$$\widehat{F_c} = -A_c^{\mathrm{t}}\,\mathcal{L}_c^{-1}\left(D_c\,P_c\,M_c^{-1}\,\Phi_c + B_c\right)$$

- The only terms depending on the time are Φ_c and B_c

Different cell subdivisions $\mathbb{P}^3$

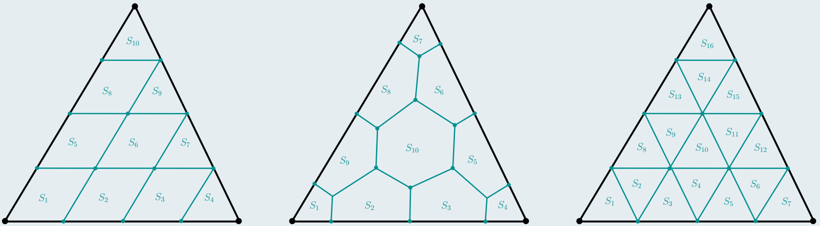
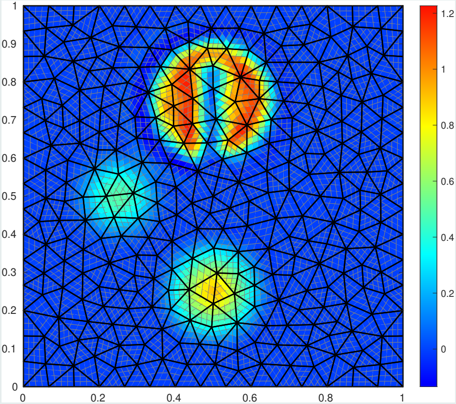


Figure : Examples of easily generalizable subdivisions for a triangle cell

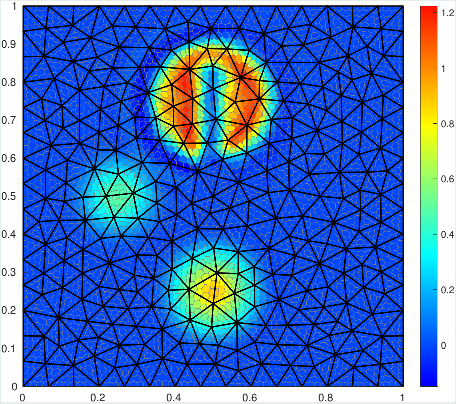
DG is DG

- **Only the functional space matters**
- **The cell subdivision has no influence on the resulting scheme**
- **Even in the case where** $N_s > N_k$

Rotation of a composite signal after one full rotation



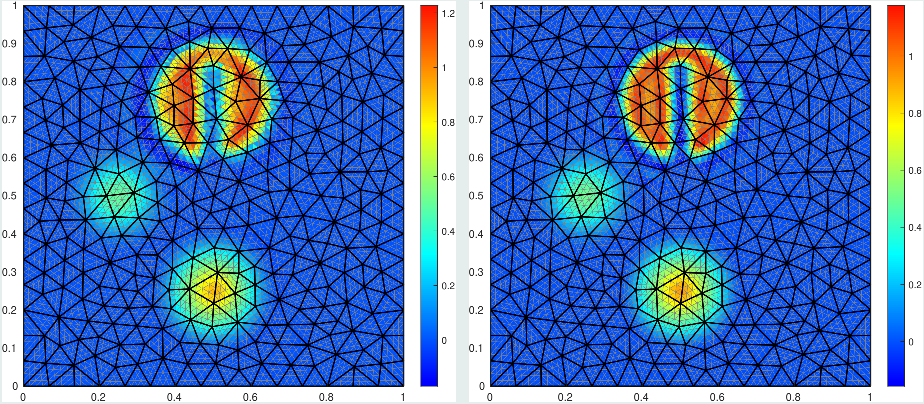
(a) Cartesian subdivision



(b) Polygonal subdivision

Figure : $\mathbb{P}^3$ reconstructed flux FV schemes on 576 cells: subcells mean values

Rotation of a composite signal after one full rotation



(a) Triangular subdivision

(b) Enriched-DG triangular subdivision

Figure : $\mathbb{P}^3$ and $\mathbb{P}^{4+\frac{1}{6}}$ reconstructed flux FV schemes on 576 cells: subcells mean values

Rotation of a composite signal after one full rotation

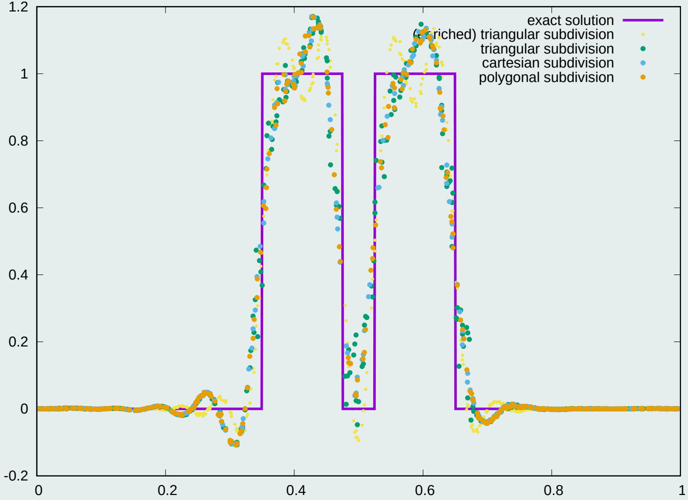
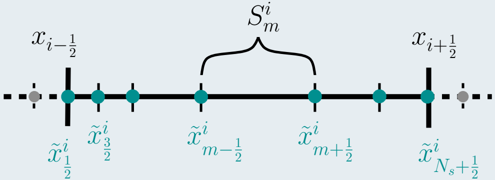


Figure : Reconstructed flux FV schemes on 576 cells: solution profiles for $y = 0.75$

Definitions $\quad \overline{u}_0^i := \overline{u}_{N_s}^{i-1} \quad$ and $\quad \overline{u}_{N_s+1}^i := \overline{u}_1^{i+1}$

- $$\frac{\mathrm{d}\,\overline{u}_m^i}{\mathrm{d}t} = -\frac{\widetilde{F}_{m+\frac{1}{2}}^i - \widetilde{F}_{m-\frac{1}{2}}^i}{|S_m^i|}$$



- $$\widetilde{F}_{m+\frac{1}{2}}^i := \mathcal{F}_{m+\frac{1}{2}}^{i,\,\mathrm{FV}} + \theta_{m+\frac{1}{2}}^i \underbrace{\left(\widehat{F}_{m+\frac{1}{2}}^i - \mathcal{F}_{m+\frac{1}{2}}^{i,\,\mathrm{FV}}\right)}_{\Delta F_{m+\frac{1}{2}}^i}$$ convex blended flux

- $\widehat{F}_{m+\frac{1}{2}}^i$ high-order reconstructed flux

- $$\mathcal{F}_{m+\frac{1}{2}}^{i,\,\mathrm{FV}} := \mathcal{F}\left(\overline{u}_m^i,\, \overline{u}_{m+1}^i\right)$$ first-order subcell numerical flux

$$= \frac{F(\overline{u}_m^i) + F(\overline{u}_{m+1}^i)}{2} - \frac{\gamma_{m+\frac{1}{2}}}{2}\left(\overline{u}_{m+1}^i - \overline{u}_m^i\right)$$ **E-flux**

Questions regarding entropy

- **Can we find the $\theta^i_{m+\frac{1}{2}}$ coefficients ensuring an entropy inequality?**
- **What do we mean by entropy inequality?**
  - for one or any entropy?
  - at the discrete or semi-discrete time level?
  - at the cells or subcells space level?
- **If we manage to ensure an entropy inequality, is it worth the effort?**
  - in terms of accuracy
  - in terms of other critical properties to ensure, as positivity for instance
- **Do we really need an entropy inequality to practically capture the entropic weak solution?**
- **If numerical diffusion is the key, how much do we need?**

Definitions

- (η, ϕ) — entropy - entropy flux
- $v(u) = \eta'(u)$ — entropy variable
- $\psi(u) = v(u)\, F(u) - \phi(u)$ — entropy potential flux
- $\phi^*(u^-, u^+) = \dfrac{\phi(u^-) + \phi(u^+)}{2} - \dfrac{\gamma}{2}\left(\eta(u^+) - \eta(u^-)\right)$
- $\eta(u^*) \leq \eta^* := \dfrac{\eta(u^-) + \eta(u^+)}{2} - \dfrac{\phi(u^+) - \phi(u^-)}{2\,\gamma}$

Subcell entropy stability at the discrete level — **for all** (η, ϕ)

- if $\Delta F_{m+\frac{1}{2}} \cdot (\overline{u}_{m+1} - \overline{u}_m) > 0$,

$$\boxed{\theta_{m+\frac{1}{2}} \leq \min\left(1,\ \frac{\left(\gamma_{m+\frac{1}{2}} - \gamma_{\text{God}}\right)\left(\overline{u}_{m+1} - \overline{u}_m\right)}{2\,\Delta F_{m+\frac{1}{2}}}\right)}$$

- γ_{God} Godunov viscosity coefficient

$\Longrightarrow$ **1st order scheme!**

Semi-discrete subcell entropy dissipation **for a given (η, ϕ)**

- if $\Delta F_{m+\frac{1}{2}} \cdot \left(v(\overline{u}_{m+1}) - v(\overline{u}_m)\right) > 0$,

$$\boxed{\theta_{m+\frac{1}{2}} \leq \min\left(1,\ \frac{\dfrac{\psi(\overline{u}_{m+1}) - \psi(\overline{u}_m)}{v(\overline{u}_{m+1}) - (\overline{u}_m)} - \mathcal{F}^{\text{FV}}_{m+\frac{1}{2}}}{\Delta F_{m+\frac{1}{2}}}\right)}$$

$\Longrightarrow$ **2nd order scheme!**

D. Kuzmin and **M. Quezada de Luna**, *Algebraic entropy fixes and convex limiting for continuous finite element discretizations of scalar hyperbolic conservation laws.* Comp. Math. Appl. Mech. Eng., 2020.

A. Rueda-Ramírez, **B. Bolm**, **D. Kuzmin** and **G. Gassner**, *Monolithic Convex Limiting for Legendre-Gauss-Lobatto Discontinuous Galerkin Spectral Element Methods.* Arxiv, 2023.

Histopolation and sub-resolution basis functions $N_s = N_k$

- Let $\{\lambda_m^c\}_m$ be the histopolation basis function such that, for $v_h^c \in \mathbb{P}^k(\omega_c)$

$$v_h^c = \sum_{m=1}^{N_k} \overline{v}_m^c \, \lambda_m^c$$

- Let $\{\varphi_m^c\}_m$ be the sub-resolution basis function such that, $\forall \psi \in \mathbb{P}^k(\omega_c)$

$$\int_{\omega_c} \varphi_m \, \psi \, \mathrm{d}V = \int_{S_m^c} \psi \, \mathrm{d}V$$

- Then, given $v_h^c \in \mathbb{P}^k(\omega_c)$, it writes

$$v_h^c = \sum_{m=1}^{N_k} \underline{v}_m^c \, \varphi_m^c$$

Orthogonality property

$$\boxed{\int_{\omega_c} \lambda_m^c \, \varphi_p^c \, \mathrm{d}V = |S_m^c| \, \delta_{mp}}$$

Semi-discrete cell entropy dissipation **for a given** (η, ϕ)

- $$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\omega_c}\eta(u_h^c)\,\mathrm{d}V = \oint_{\omega_c} v(u_h^c)\,\partial_t u_h^c\,\mathrm{d}V = \int_{\omega_c} v_h^c\,\partial_t u_h^c\,\mathrm{d}V \equiv \Delta\eta_c$$

- $v_h^c = \sum_{m=1}^{N_k} \underline{v}_m^c\,\varphi_m^c$ L^2 projection of $v(u_h^c)$ onto $\mathbb{P}^k$

- $$\Delta\eta_c = \int_{\omega_c}\left(\sum_{m=1}^{N_k}\underline{v}_m^c\,\varphi_m^c\right)\left(\sum_{m=1}^{N_k}\frac{\mathrm{d}\,\overline{u}_m^c}{\mathrm{d}t}\,\lambda_m^c\right)\mathrm{d}V = \sum_{m=1}^{N_k}|S_m^c|\,\underline{v}_m^c\,\frac{\mathrm{d}\,\overline{u}_m^c}{\mathrm{d}t}$$

For sake of simplicity, let us consider 1D $N_k = k+1$

- $$\Delta\eta_i = -\sum_{m=1}^{k+1}\underline{v}_m^i\left(\widetilde{F}_{m+\frac{1}{2}} - \widetilde{F}_{m-\frac{1}{2}}\right) = \mathbf{A}_{vol} + \mathbf{A}_{bdr}$$

- $$\mathbf{A}_{vol} = \sum_{m=1}^{k+1}\left(\underline{v}_{m+1}^i - \underline{v}_m^i\right)\widetilde{F}_{m+\frac{1}{2}} + \left(\underline{v}_1^i - v\left(u_h^i(x_{i-\frac{1}{2}})\right)\right)\theta_{\frac{1}{2}}^i\,\widehat{F}_{\frac{1}{2}}^i + \left(v\left(u_h^i(x_{i+\frac{1}{2}})\right) - \underline{v}_{k+1}^i\right)\theta_{k+\frac{3}{2}}^i\,\widehat{F}_{k+\frac{3}{2}}^i$$

Boundary entropy contribution

- $\mathcal{F}^{\mathrm{FV}}_{i-\frac{1}{2}} \equiv \mathcal{F}^{i,\mathrm{FV}}_{\frac{1}{2}}, \quad \mathcal{F}^{\mathrm{FV}}_{i+\frac{1}{2}} \equiv \mathcal{F}^{i,\mathrm{FV}}_{k+\frac{3}{2}}, \quad \mathcal{F}^{\mathrm{DG}}_{i-\frac{1}{2}} \equiv \widehat{F^i_{\frac{1}{2}}}, \quad \mathcal{F}^{\mathrm{DG}}_{i+\frac{1}{2}} \equiv \widehat{F^i_{k+\frac{3}{2}}}$
- $\theta_{i+\frac{1}{2}} \equiv \theta^i_{k+\frac{3}{2}} = \theta^{i+1}_{\frac{1}{2}}$
- $v^{\mp}_{i\pm\frac{1}{2}} \equiv v\big(u^i_h(x_{i\pm\frac{1}{2}})\big)$
- $$\mathbf{A}_{bdr} = \underline{v}^i_1 \left(1 - \theta_{i-\frac{1}{2}}\right) \mathcal{F}^{\mathrm{FV}}_{i-\frac{1}{2}} + v^+_{i-\frac{1}{2}}\, \theta_{i-\frac{1}{2}}\, \mathcal{F}^{\mathrm{DG}}_{i-\frac{1}{2}}$$
$$-\underline{v}^i_{k+1} \left(1 - \theta_{i+\frac{1}{2}}\right) \mathcal{F}^{\mathrm{FV}}_{i+\frac{1}{2}} - v^-_{i+\frac{1}{2}}\, \theta_{i+\frac{1}{2}}\, \mathcal{F}^{\mathrm{DG}}_{i+\frac{1}{2}}$$

Semi-discrete cell entropy stability **for a given** (η, ϕ)

- $\mathcal{F}^{\mathrm{FV}}_{m+\frac{1}{2}} := \mathcal{F}\big(u(\underline{v}_{m+1}), u(\underline{v}_m)\big)$ modified FV numerical flux
- A sufficient condition to entropy stability writes as follows

$$\mathbf{A}_{vol} \leq \theta_{i-\frac{1}{2}} \left(\psi(\underline{v}^i_1) - \psi(v^+_{i-\frac{1}{2}})\right) + \theta_{i+\frac{1}{2}} \left(\psi(v^-_{i+\frac{1}{2}}) - \psi(\underline{v}^i_{k+1})\right) + \psi(\underline{v}^i_{k+1}) - \psi(\underline{v}^i_1)$$

Y. Lin and J. Chan, *High order entropy stable discontinuous Galerkin spectral element methods through subcell limiting.* JCP, 2024.

Proof

- $$\Delta\eta_i \leq \left(1-\theta_{i-\frac{1}{2}}\right)\left(\underline{v}_1^i\,\mathcal{F}_{i-\frac{1}{2}}^{\mathrm{FV}} - \psi\left(\underline{v}_1^i\right)\right) + \theta_{i-\frac{1}{2}}\left(v_{i-\frac{1}{2}}^+\,\mathcal{F}_{i-\frac{1}{2}}^{\mathrm{DG}} - \psi\left(v_{i-\frac{1}{2}}^+\right)\right)$$
$$-\left(1-\theta_{i+\frac{1}{2}}\right)\left(\underline{v}_{k+1}^i\,\mathcal{F}_{i+\frac{1}{2}}^{\mathrm{FV}} - \psi\left(\underline{v}_{k+1}^i\right)\right) - \theta_{i+\frac{1}{2}}\left(v_{i+\frac{1}{2}}^-\,\mathcal{F}_{i+\frac{1}{2}}^{\mathrm{DG}} - \psi\left(v_{i+\frac{1}{2}}^-\right)\right)$$

- Given $(v_L,\, v_R)$, if $\exists\ \mathrm{D} \geq 0$ such that

$$\mathcal{F}\Big(u(v_L),\, u(v_R)\Big) = \frac{\psi(v_R)-\psi(v_L)}{v_R - v_L} - \frac{\mathrm{D}}{2}\,(v_R - v_L)$$

then

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\omega_i}\eta(u_h^i)\,\mathrm{d}V \leq -\left(\widetilde{\phi_{i+\frac{1}{2}}} - \widetilde{\phi_{i-\frac{1}{2}}}\right)}$$

- $$\widetilde{\phi_{i+\frac{1}{2}}} = \left(1-\theta_{i+\frac{1}{2}}\right)\phi^*\left(\underline{v}_{k+1}^i,\, \underline{v}_1^{i+1}\right) + \theta_{i+\frac{1}{2}}\ \phi^*\left(v_{i+\frac{1}{2}}^-,\, v_{i+\frac{1}{2}}^+\right)$$

- $$\phi^*(v_L,\, v_R) = \frac{v_L+v_R}{2}\,\mathcal{F}\Big(u(v_L),\, u(v_R)\Big) - \frac{\psi(v_L)+\psi(v_R)}{2}$$

Knapsack problem

- The sufficient condition rewrites as

$$\boxed{\boldsymbol{a} \,.\, \boldsymbol{\Theta} \leq b}$$

- $\boldsymbol{\Theta} = \left(\theta^i_{\frac{1}{2}}, \ldots, \theta^i_{k+\frac{3}{2}}\right)^{\mathrm{t}}$
- $\boldsymbol{a} = \left(a^i_{\frac{1}{2}}, \ldots, a^i_{k+\frac{3}{2}}\right)^{\mathrm{t}}$ defined as

$$\begin{cases} a_{\frac{1}{2}} = \left(\underline{v}^i_1 - v^+_{i-\frac{1}{2}}\right) \mathcal{F}^{\mathrm{DG}}_{i-\frac{1}{2}} - \Big(\psi\big(\underline{v}^i_1\big) - \psi\big(v^+_{i-\frac{1}{2}}\big)\Big), \\ a_{m+\frac{1}{2}} = \left(\underline{v}^i_{m+1} - \underline{v}^i_m\right) \left(\widehat{F_{m+\frac{1}{2}}} - \mathcal{F}^{\mathrm{FV}}_{m+\frac{1}{2}}\right), \qquad m \in [\![1, k]\!] \\ a_{k+\frac{3}{2}} = \left(v^-_{i+\frac{1}{2}} - \underline{v}^i_{k+1}\right) \mathcal{F}^{\mathrm{DG}}_{i+\frac{1}{2}} - \Big(\psi\big(v^-_{i+\frac{1}{2}}\big) - \psi\big(\underline{v}^i_{k+1}\big)\Big), \end{cases}$$

- $b = \psi\big(\underline{v}^i_{k+1}\big) - \psi\big(\underline{v}^i_1\big) - \sum_{m=1}^{k+1} \left(\underline{v}^i_{m+1} - \underline{v}^i_m\right) \mathcal{F}^{\mathrm{FV}}_{m+\frac{1}{2}}$
- **Because $b \geq 0$ the Knapsack problem is indeed solvable**

Greedy algorithm

- Find $\mathbf{0} \leq \mathbf{\Theta} \leq \mathbf{\Theta}_C \leq \mathbf{1}$ maximizing $\sum_{m=1}^{k+1} \theta_{m+\frac{1}{2}}$ such that $\boxed{\boldsymbol{a} \,.\, \mathbf{\Theta} \leq b}$
- $\mathbf{\Theta}_C = \left(\theta^C_{\frac{1}{2}}, \ldots, \theta^C_{k+\frac{3}{2}}\right)^{\mathrm{t}}$ is a given supplementary constraint

High-order accuracy preservation

- Let us consider u a smooth exact solution
- $u_h^i = u + O(\Delta x^{k+1})$ $(k+1)^{\text{th}}$-order approximation
- Then, we have that

$$\boxed{\boldsymbol{a} \,.\, \mathbf{1} - b = O(\Delta x^{k+2})}$$

- This implies that

$$\boxed{\widetilde{F^i_{m+\frac{1}{2}}} = F\big(u(x^i_{m+\frac{1}{2}})\big) + O(\Delta x^{k+1})}$$

Linear advection of a composite signal $\eta(u) = \frac{1}{2}u^2$

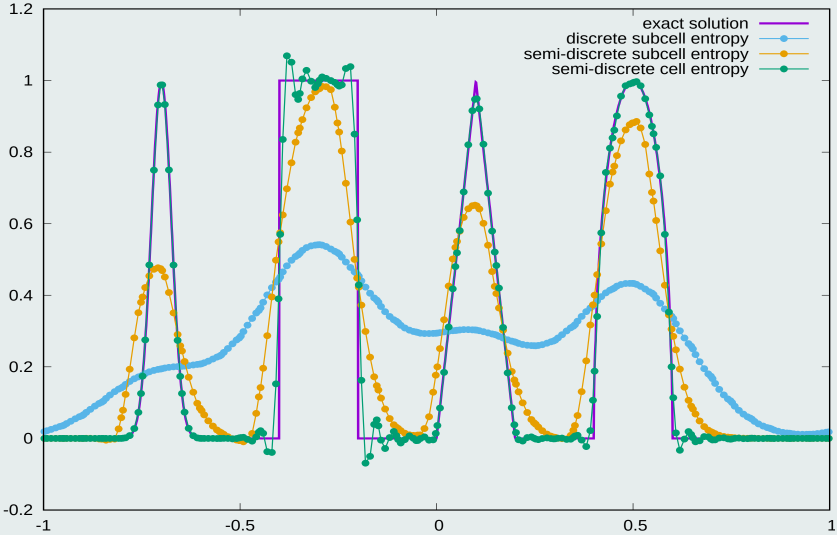


Figure : $\mathbb{P}^5$-DG/FV solutions on 40 cells: submean values

Linear advection of a composite signal $\eta(u) = |u - k_e|^{1+\epsilon} \backslash (1 + \epsilon)$

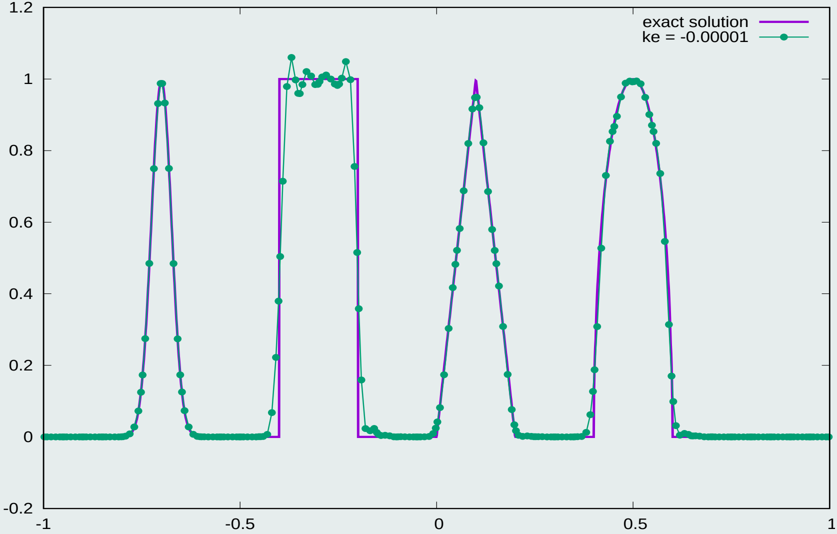


Figure : $\mathbb{P}^5$-DG/FV submean values on 40 cells: $\epsilon = 0.25$ and k_e =-1.D-5

Linear advection of a composite signal $\eta(u) = |u - k_e|^{1+\epsilon} \backslash (1+\epsilon)$

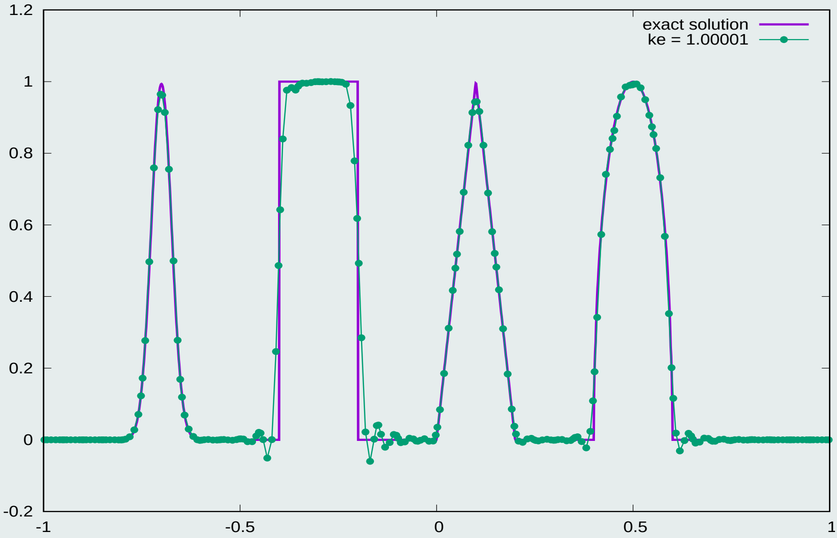


Figure : $\mathbb{P}^5$-DG/FV submean values on 40 cells: $\epsilon = 0.25$ and $k_e = 1 + 1.D\text{-}5$

Non-linear non-convex flux Buckley case — 80 cells

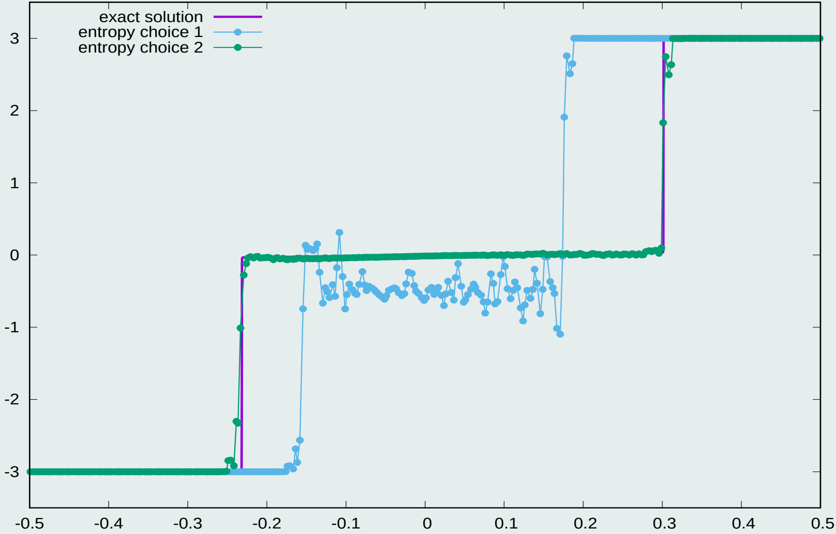


Figure : $\mathbb{P}^3$-DG/FV submean values: $\eta_1(u) = \frac{1}{2}u^2$ and $\eta_2(u) = \int \mathrm{atan}(20u)\,\mathrm{d}u$

Non-linear non-convex flux Buckley case — 80 cells

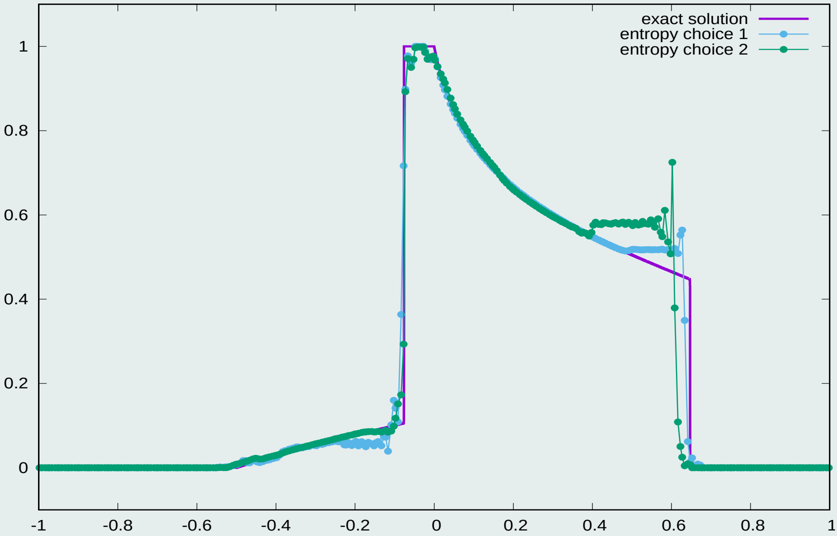
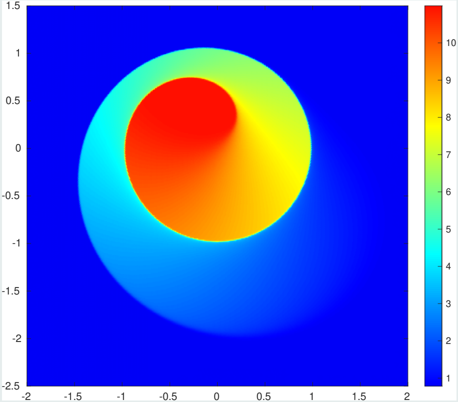
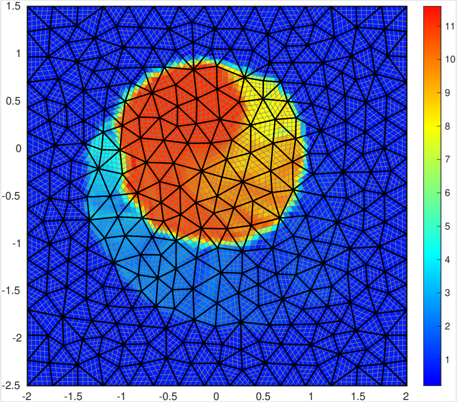


Figure : $\mathbb{P}^3$-DG/FV submean values: $\eta_1(u) = \frac{1}{2}u^2$ and $\eta_2(u) = \int \text{atan}(20u)\,\mathrm{d}u$

KPP non-convex flux problem $\eta(u) = \frac{1}{2}u^2$



(a) 1^{th}-order FV on 209184 cells



(b) $\mathbb{P}^4$-DG/FV on 576 cells

Figure : $\mathbb{P}^4$-DG/FV entropic scheme: non-entropic solution

Questions regarding entropy $\longrightarrow$ pieces of answer

- **Can we find $\theta^i_{m+\frac{1}{2}}$ coefficients ensuring an entropy inequality?**

  $\hookrightarrow$ **Yes!**

- **What do we mean by entropy inequality, and is it worth the effort?**
  - for any entropy, at the discrete time level and for any subcell

    $\hookrightarrow$ **1^{st}-order**
  - for a given entropy, at the semi-discrete time level for any subcells

    $\hookrightarrow$ **2^{nd}-order**
  - for a given entropy, at the semi-discrete time level for any cells

    $\hookrightarrow$ **$(k+1)^{\text{th}}$-order**

    $\hookrightarrow$ $\mathcal{F}^{\text{FV}}_{m+\frac{1}{2}} := \mathcal{F}\big(\boldsymbol{u}(\underline{\boldsymbol{v}}_{m+1}), \boldsymbol{u}(\underline{\boldsymbol{v}}_m)\big)$ ⚠

- **Do we need entropy stability or just *“enough”* numerical diffusion?**

  $\hookrightarrow$ **Unclear...** $\Longrightarrow$ **GMP and LMP + relaxation**

Global maximum principle $\overline{u}_m^{n+1} \in [\alpha, \beta]$

$$\theta_{m+\frac{1}{2}} \leq \min\left(1, \underbrace{\left|\frac{\gamma_{m+\frac{1}{2}}}{\Delta F_{m+\frac{1}{2}}}\right|}_{D_{m+\frac{1}{2}}} \min\left(\beta - u^*_{m+\frac{1}{2}},\, u^*_{m+\frac{1}{2}} - \alpha\right)\right)$$

Local maximum principle $\overline{u}_m^{n+1} \in \mathrm{I}\left(\overline{u}_{m-1}^n, \overline{u}_m^n, \overline{u}_{m+1}^n\right)$

- $\widetilde{u_{m+\frac{1}{2}}}^{-} \in [\alpha_m, \beta_m] := \mathrm{I}\left(\overline{u}_{m-1}^n, \overline{u}_m^n, \overline{u}_{m+1}^n\right)$
- $\widetilde{u_{m+\frac{1}{2}}}^{+} \in [\alpha_{m+1}, \beta_{m+1}] := \mathrm{I}\left(\overline{u}_m^n, \overline{u}_{m+1}^n, \overline{u}_{m+2}^n\right)$

$$\theta_{m+\frac{1}{2}} \leq \min\left(1, \left|D_{m+\frac{1}{2}}\right| \begin{cases} \min\left(\beta_{m+1} - u^*_{m+\frac{1}{2}},\, u^*_{m+\frac{1}{2}} - \alpha_m\right) & \text{if } \Delta F_{m+\frac{1}{2}} > 0 \\ \min\left(\beta_m - u^*_{m+\frac{1}{2}},\, u^*_{m+\frac{1}{2}} - \alpha_{m+1}\right) & \text{if } \Delta F_{m+\frac{1}{2}} < 0 \end{cases}\right)$$

- Smooth extrema relaxation to preserve accuracy

Linear advection of a composite signal — 40 cells

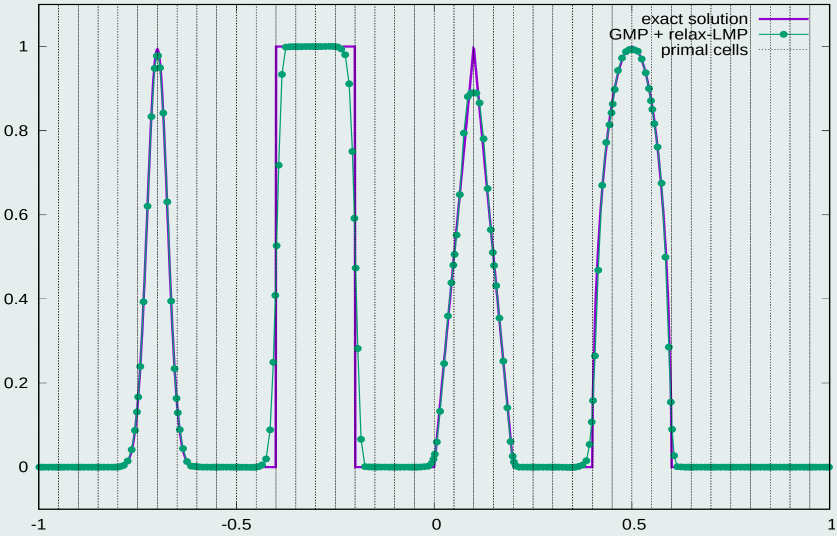


Figure : $\mathbb{P}^6$-DG/FV with GMP and relaxed-LMP: submean values

Linear advection of a composite signal — 40 cells

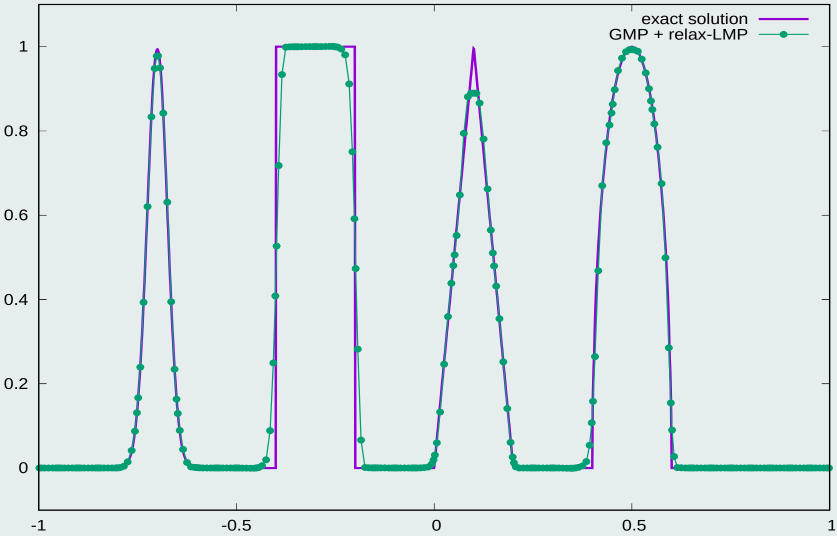


Figure : $\mathbb{P}^6$-DG/FV with GMP and relaxed-LMP: submean values

Non-linear non-convex flux Buckley case 40 cells

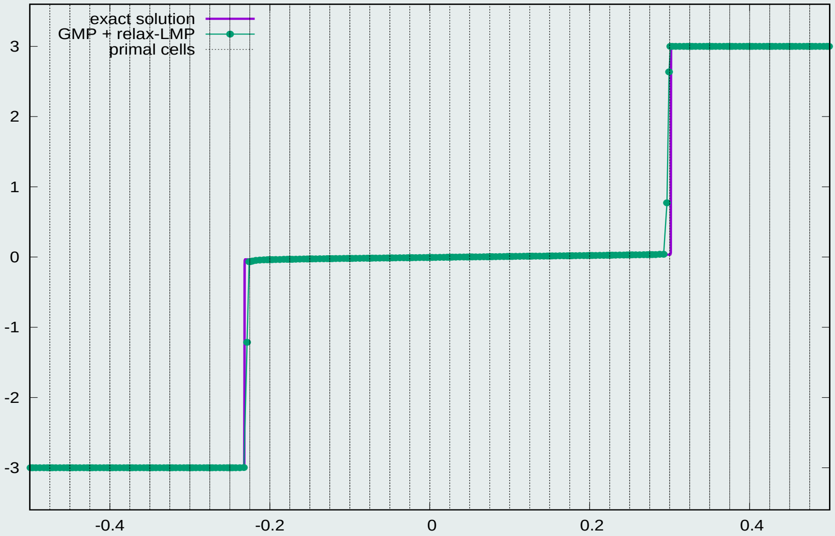


Figure : $\mathbb{P}^6$-DG/FV with GMP and relaxed-LMP: submean values

Non-linear non-convex flux Buckley case 40 cells

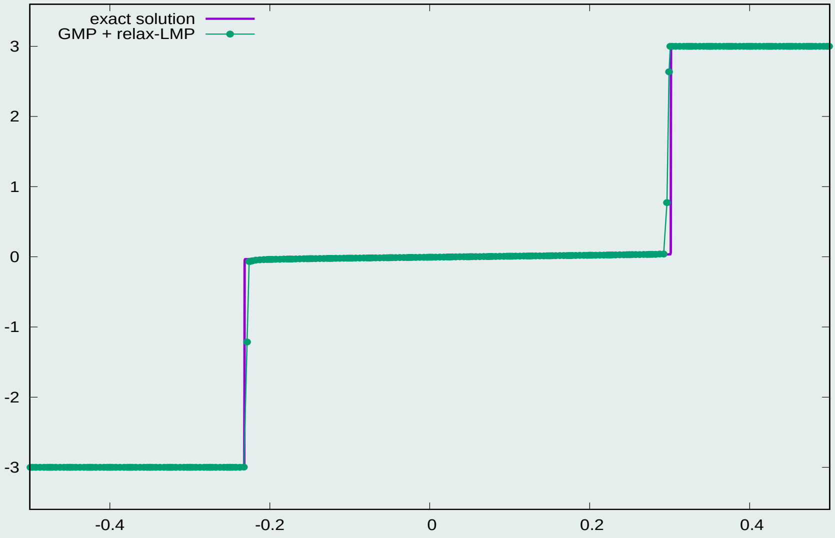


Figure : $\mathbb{P}^6$-DG/FV with GMP and relaxed-LMP: submean values

Non-linear non-convex flux Buckley case 40 cells

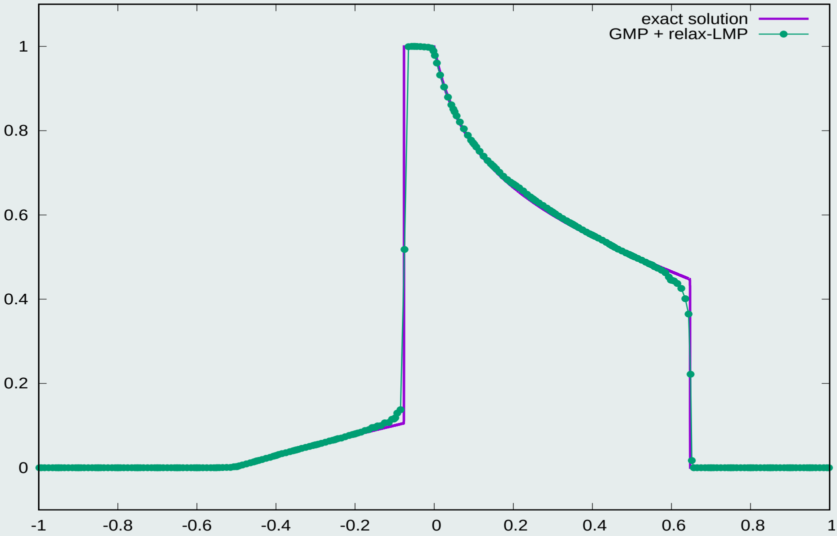
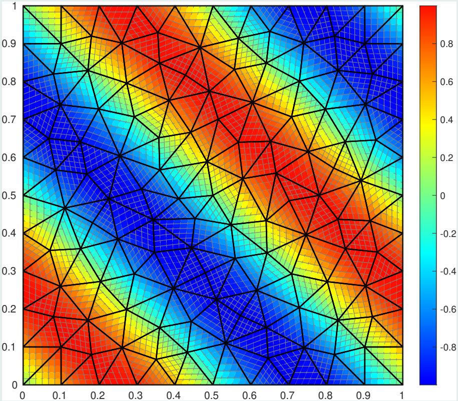
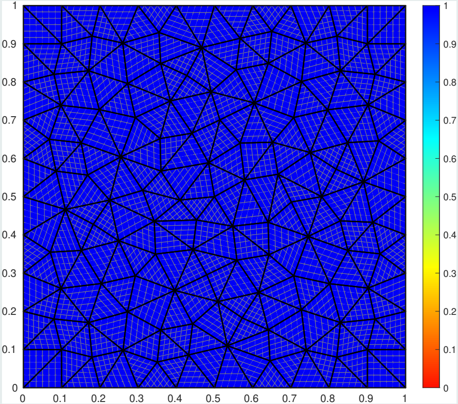


Figure : $\mathbb{P}^6$-DG/FV with GMP and relaxed-LMP: submean values

Burgers equation $u_0(x, y) = \sin(2\pi\,(x + y))$



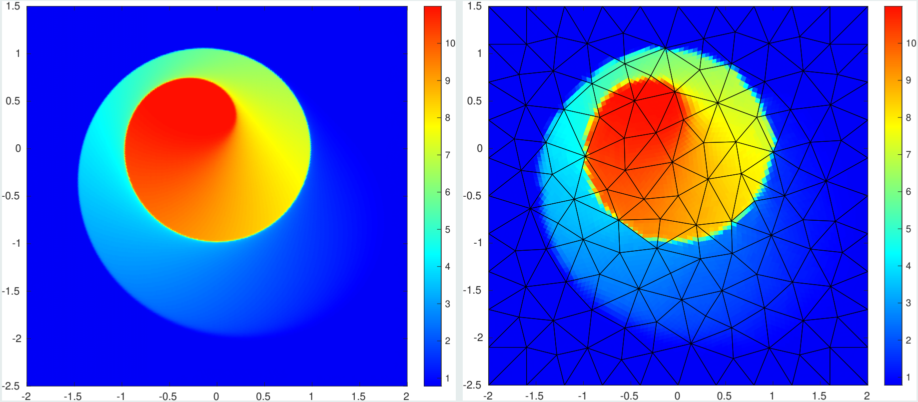
(a) Solution submean values



(b) Blending coefficients

Figure : $\mathbb{P}^5$-DG/FV scheme with GMP and relaxed-LMP on 242 cells

KPP non-convex flux problem



(a) 1^{th}-order FV on 209184 cells

(b) $\mathbb{P}^7$-DG/FV on 272 cells

Figure : $\mathbb{P}^7$-DG/FV scheme with GMP and relaxed-LMP

Non-linear Euler compressible gas dynamics equations

- $\partial_t \boldsymbol{V} + \nabla_x \cdot \boldsymbol{F}(\boldsymbol{V}) = \mathbf{0}$

- $\boldsymbol{V} = \begin{pmatrix} \rho \\ \boldsymbol{q} \\ E \end{pmatrix}$ conservative variables

- $\boldsymbol{F}(\boldsymbol{V}) = \begin{pmatrix} \boldsymbol{q} \\ \dfrac{\boldsymbol{q} \otimes \boldsymbol{q}}{\rho} + p\, I_d \\ (E + p)\, \dfrac{\boldsymbol{q}}{\rho} \end{pmatrix}$ flux function

- $p := p(\boldsymbol{V}) = (\gamma - 1)\left(E - \dfrac{1}{2}\dfrac{\|\boldsymbol{q}\|^2}{\rho}\right)$ equation of state

Monolithic subcell DG/FV scheme property

- Positivity of the density and internal energy, at the subcell scale

Definitions

- $\widetilde{\boldsymbol{F}_{m+\frac{1}{2}}} := \mathcal{F}^{\text{FV}}_{m+\frac{1}{2}} + \Theta_{m+\frac{1}{2}} \underbrace{\left(\widehat{\boldsymbol{F}_{m+\frac{1}{2}}} - \mathcal{F}^{\text{FV}}_{m+\frac{1}{2}}\right)}_{\Delta \boldsymbol{F}_{m+\frac{1}{2}}}$ convex blended flux
- $\Theta_{m+\frac{1}{2}} = \begin{pmatrix} \theta^{\rho}_{m+\frac{1}{2}} & 0 & 0 \\ 0 & \theta^{q}_{m+\frac{1}{2}} & 0 \\ 0 & 0 & \theta^{E}_{m+\frac{1}{2}} \end{pmatrix}$
- $\mathcal{F}^{\text{FV}}_{m+\frac{1}{2}} := \mathcal{F}\left(\overline{u}_m,\, \overline{u}_{m+1}\right)$ Global L-F, Rusanov, HLL(C), ...

Positivity of the density

- $\theta^{\rho}_{m+\frac{1}{2}} = \theta^{\rho\,(1)}_{m+\frac{1}{2}}\, \theta^{(2)}_{m+\frac{1}{2}}$

$$\boxed{\theta^{\rho\,(1)}_{m+\frac{1}{2}} \leq \min\left(1,\, \left|\frac{\gamma_{m+\frac{1}{2}}}{\Delta F^{\rho}_{m+\frac{1}{2}}}\right| \rho^{*}_{m+\frac{1}{2}}\right)}$$

Positivity of the internal energy

- $A_{m+\frac{1}{2}} = \dfrac{1}{(\gamma_{m+\frac{1}{2}})^2} \left(\dfrac{1}{2} \left(\Delta F^q_{m+\frac{1}{2}} \right)^2 - \theta^{\rho\,(1)}_{m+\frac{1}{2}} \Delta F^\rho_{m+\frac{1}{2}} \, \Delta F^E_{m+\frac{1}{2}} \right)$
- $B_{m+\frac{1}{2}} = \dfrac{1}{\gamma_{m+\frac{1}{2}}} \left(q^*_{m+\frac{1}{2}} \, \Delta F^q_{m+\frac{1}{2}} - \rho^*_{m+\frac{1}{2}} \, \Delta F^E_{m+\frac{1}{2}} - \theta^{\rho\,(1)}_{m+\frac{1}{2}} \, E^*_{m+\frac{1}{2}} \, \Delta F^\rho_{m+\frac{1}{2}} \right)$
- $M_{m+\frac{1}{2}} = \rho^*_{m+\frac{1}{2}} \, E^*_{m+\frac{1}{2}} - \frac{1}{2} \left(q^*_{m+\frac{1}{2}} \right)^2$

$$\boxed{\theta^{(2)}_{m+\frac{1}{2}} \leq \min \left(1, \frac{M_{m+\frac{1}{2}}}{\left| B_{m+\frac{1}{2}} \right| + \max \left(0, A_{m+\frac{1}{2}} \right)} \right)}$$

- $\theta^\rho_{m+\frac{1}{2}} = \theta^{\rho\,(1)}_{m+\frac{1}{2}} \, \theta^{(2)}_{m+\frac{1}{2}}, \quad \theta^q_{m+\frac{1}{2}} = \theta^{(2)}_{m+\frac{1}{2}}, \quad \theta^E_{m+\frac{1}{2}} = \theta^{(2)}_{m+\frac{1}{2}}$

A. Rueda-Ramírez, B. Bolm, D. Kuzmin and **G. Gassner**, *Monolithic Convex Limiting for Legendre-Gauss-Lobatto Discontinuous Galerkin Spectral Element Methods.* Arxiv, 2023.

LMP $\overline{v}_m^{n+1} \in \mathrm{I}\left(\overline{v}_{m-1}^n,\, v_{m-\frac{1}{2}}^*,\, \overline{v}_m^n,\, v_{m+\frac{1}{2}}^*,\, \overline{v}_{m+1}^n\right)$

- $v \in \{\rho,\, q,\, E\}$ conservative variable
- $\widetilde{v_{m+\frac{1}{2}}}^{-} \in [\alpha_m,\, \beta_m] := \mathrm{I}\left(\overline{v}_{m-1}^n,\, v_{m-\frac{1}{2}}^*,\, \overline{v}_m^n,\, v_{m+\frac{1}{2}}^*,\, \overline{v}_{m+1}^n\right)$
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$$\theta_{m+\frac{1}{2}} \leq \min\left(1,\, \left|D_{m+\frac{1}{2}}\right| \begin{cases} \min\left(\beta_{m+1} - v_{m+\frac{1}{2}}^*,\, v_{m+\frac{1}{2}}^* - \alpha_m\right) & \text{if } \Delta F_{m+\frac{1}{2}} > 0 \\ \min\left(\beta_m - v_{m+\frac{1}{2}}^*,\, v_{m+\frac{1}{2}}^* - \alpha_{m+1}\right) & \text{if } \Delta F_{m+\frac{1}{2}} < 0 \end{cases}\right)$$

- Smooth extrema relaxation to preserve accuracy

Entropy inequalities

- The same techniques apply here $\Longrightarrow$ **But is that really needed?**

Sod shock tube test case 10 cells

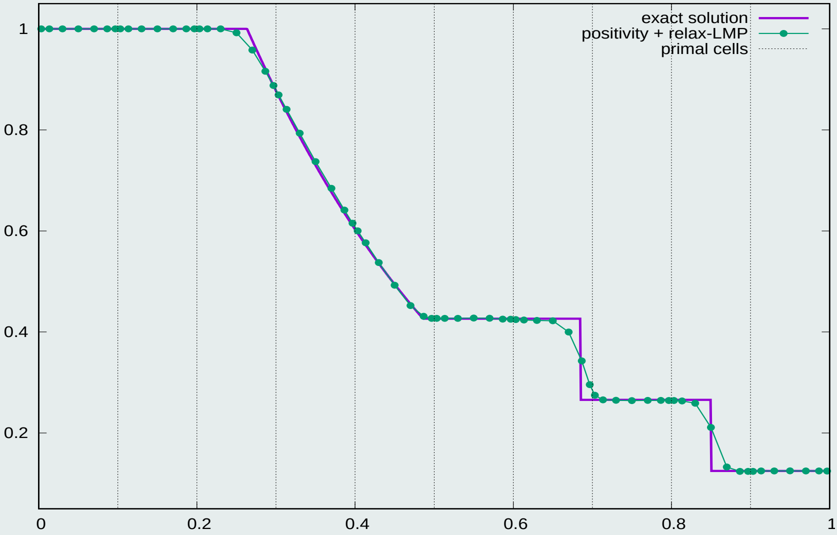


Figure : $\mathbb{P}^6$-DG/FV scheme with GMP and relaxed-LMP: submean values

Double rarefaction test case 10 cells

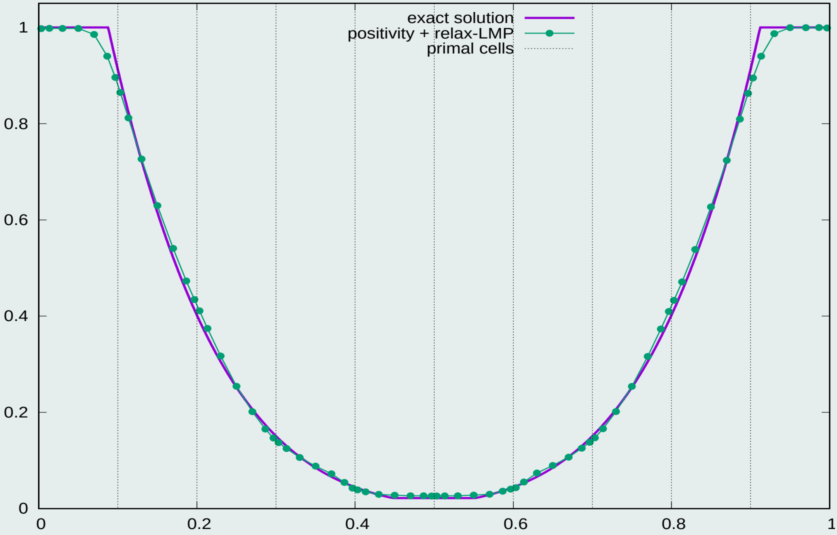


Figure : $\mathbb{P}^6$-DG/FV scheme with GMP and relaxed-LMP: submean values

Shock acoustic-wave interaction test case 200 cells

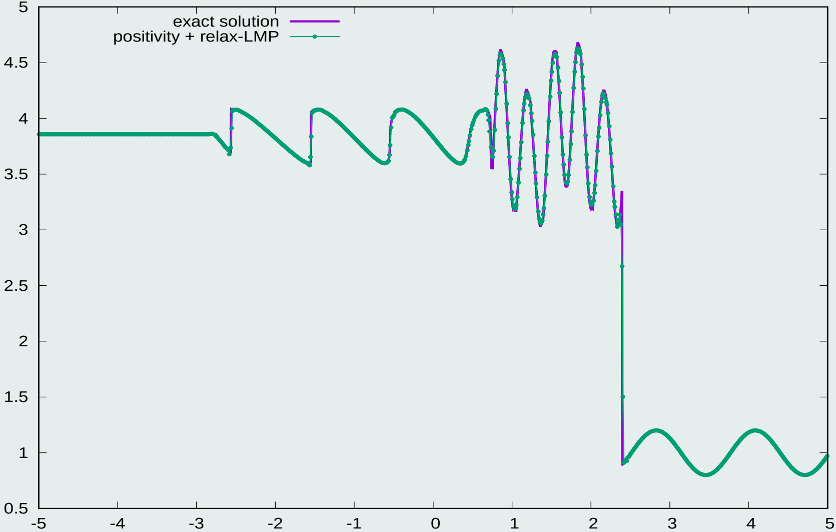
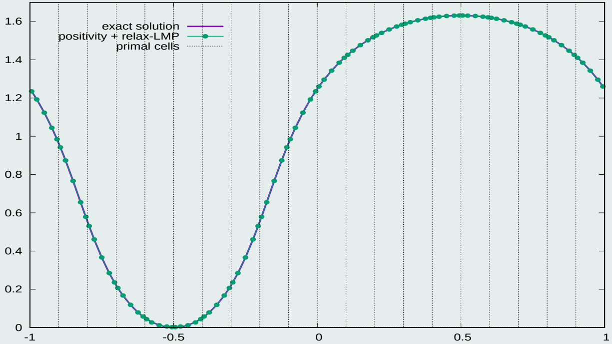


Figure : $\mathbb{P}^3$-DG/FV scheme with GMP and relaxed-LMP: HLL-C numerical flux

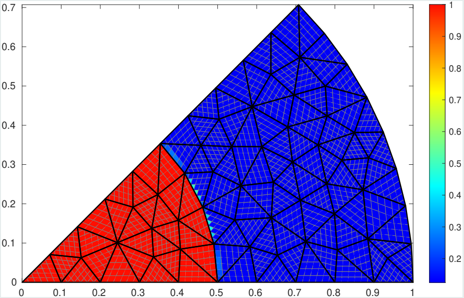
Smooth isentropic solution $\rho_0 = 1 + 0.9999999 \sin(2\pi x)$



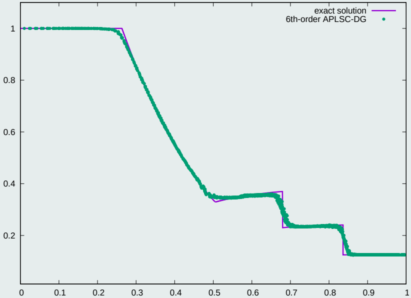
	L_1		L_2	
h	$E^h_{L_1}$	$q^h_{L_1}$	$E^h_{L_2}$	$q^h_{L_2}$
$\frac{1}{20}$	1.54E-5	4.01	2.04E-5	3.82
$\frac{1}{40}$	9.57E-7	4.89	1.45E-6	4.85
$\frac{1}{80}$	3.22E-8	4.84	5.00E-8	4.87
$\frac{1}{160}$	1.12E-9	-	1.71E-9	-

Table: Convergence rates computed on the pressure with a 5th-order DG/FV scheme

Sod shock tube problem in cylindrical geometry



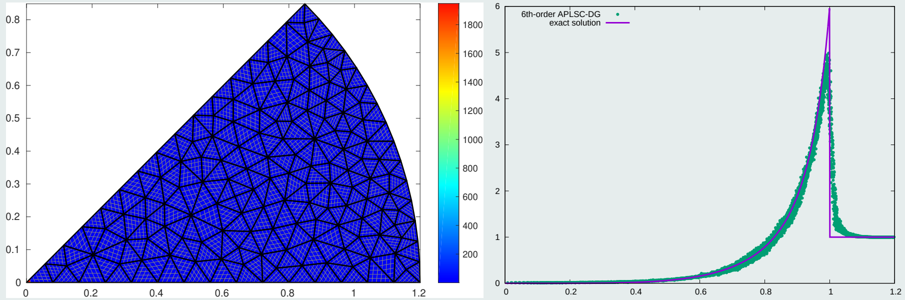
(a) Density map



(b) Density profile

Figure : 6th-order APLSC-DG with GMP and relaxed-LMP on a 110 cells mesh

Sedov point blast problem in cylindrical geometry



(a) Energy map

(b) Density profile

Figure : 6th-order APLSC-DG on a 271 cells mesh at $t = 1$

Monolithic local subcell DG/FV scheme

- Reformulate DG schemes as subgrid FV-like schemes:
  - regardless the type of mesh used
  - regardless the space dimension (***in theory***. . .)
  - regardless the cell subdivision ($N_s \geq N_k$)
- Combine high-order reconstructed fluxes and 1st-order FV fluxes
  - ensuring a maximum or positivity preserving principle at the subcell scale
  - ensuring different entropy stability inequalities
  - reducing significantly the apparition of spurious oscillations
  - preserving the very accurate subcell resolution of DG schemes

Questions

- Is an entropy inequality for one entropy enough?

  $\hookrightarrow$ **Generally, no**
- Is entropy inequality absolutely needed?

  $\hookrightarrow$ **Maybe not** $\Longrightarrow$ **GMP + relaxed LMP**

Monolithic local subcell DG/FV scheme

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Articles on this topic

- **F. Vilar**, *A Posteriori Correction of High-Order DG Scheme through Subcell Finite Volume Formulation and Flux Reconstruction.* JCP, 387:245-279, 2018.
- **A. Haidar**, **F. Marche** and **F. Vilar**, *A posteriori Finite-Volume local subcell correction of high-order discontinuous Galerkin schemes for the nonlinear shallow-water equations*. JCP, 452:110902, 2022.
- **F. Vilar** and **R. Abgrall**, *A posteriori local subcell correction of DG schemes through Finite Volume reformulation on unstructured grids.* SIAM SISC, 46(2), 2024.
- **A. Haidar**, **F. Marche** and **F. Vilar**, *Free-boundary problems for wave-structure interactions in shallow-water: DG-ALE description and local subcell correction*. JSC, 98(45), 2024.
- **A. Haidar**, **F. Marche** and **F. Vilar**, *A robust DG-ALE formulation for nonlinear shallow-water interactions with a floating object*. JSC, under review, 2024.

Linear advection of a composite signal $\eta(u) = |u - k_e|^{1+\epsilon} \backslash (1 + \epsilon)$

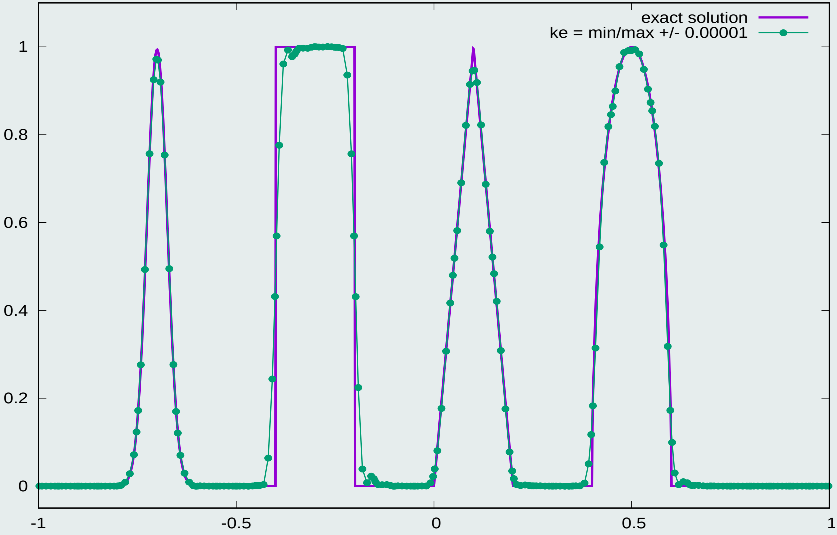

Figure : $\mathbb{P}^5$-DG/FV on 40 cells: $\epsilon = 0.25$ and k_e =min\max $\mp$ 1.D-5